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2005 J. Phys. A: Math. Gen. 38 R133

(http://iopscience.iop.org/0305-4470/38/18/R01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) R133–R193 doi:10.1088/0305-4470/38/18/R01

**TOPICAL REVIEW**

# Ageing properties of critical systems

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Received 11 October 2004, in final form 15 March 2005
Published 18 April 2005
Online at stacks.iop.org/JPhysA/38/R133

**Abstract**

In the past few years, systems with slow dynamics have attracted considerable theoretical and experimental interest. Ageing phenomena are observed during this everlasting non-equilibrium evolution. A simple instance of such a behaviour is provided by the dynamics that takes place when a system is quenched from its high-temperature phase to the critical point. The aim of this review is to summarize the various numerical and analytical results that have been recently obtained for this case. Particular emphasis is put on the field-theoretical methods that can be used to provide analytical predictions for the relevant dynamical quantities. Fluctuation–dissipation relations are discussed and in particular the concept of fluctuation–dissipation ratio (FDR) is reviewed, emphasizing its connection with the definition of a possible effective temperature. The renormalization-group approach to critical dynamics is summarized and the scaling forms of the time-dependent non-equilibrium correlation and response functions of a generic observable are discussed. From them, the universality of the associated FDR follows as an amplitude ratio. It is then possible to provide predictions for ageing quantities in a variety of different models. In particular, the results for models A, B and C dynamics of the $O(N)$ Ginzburg–Landau Hamiltonian, and model A dynamics of the weakly dilute Ising magnet and of the $\varphi^3$ theory are reviewed and compared with the available numerical results and exact solutions. The effect of a planar surface on the ageing behaviour of model A dynamics is also addressed within the mean-field approximation.

PACS numbers: 05.10.Cc, 05.10.Gg, 64.60.Ht, 75.40.Gb

  

## Contents



## Nomenclature and notation

| | |
|---|---|
| A, B, C, ... J | dynamical universality classes |
| FD | fluctuation–dissipation |
| FDR | fluctuation–dissipation ratio |
| FDT | fluctuation–dissipation theorem |
| FT | field theory |
| IR | infra-red |
| MC | Monte Carlo (simulation) |

| | |
|---|---|
| MF | mean field |
| MS | minimal subtraction |
| OT | ordinary transition (for surface critical behaviour) |
| RG | renormalization group |
| RIM | random Ising model |
| $T_c$ | critical temperature |
| $T_{\rm eff}$ | effective temperature |
| SDE | short-distance expansion |
| SpT | special transition (for surface critical behaviour) |
| $t, s$ | times with $t > s$, if not differently specified |
| TTI | time-translation invariance |
| TRS | time-reversal symmetry |
| UV | ultraviolet |
| $(\mathrm{d}q)$ | $\mathrm{d}^d q/(2\pi)^d$ |

## 1. Introduction

Understanding the non-equilibrium dynamics of physical systems is currently one of the most challenging problems in statistical physics. Equilibrium statistical mechanics has been probably one of the most important achievements during the last century. On the other hand, in nature equilibrium is more an exception rather than a rule: real systems may persist out-of-equilibrium for several reasons, such as external driving forces, very slow relaxation etc. Many efforts are currently aiming at achieving a coherent theoretical picture of a prototype of such non-equilibrium system: glassy systems, such as structural glasses and spin glasses. One of the most striking features of glassy systems is a dramatic slowing down of relaxational processes. After a perturbation (e.g., a sudden temperature change) the slow dynamics manifests itself in one-time quantities, such as the energy density, through some slow-decaying dependence on the time elapsed from the perturbation. However, even if these quantities get very close to an asymptotic value, this does not mean that the system is reaching some stationary or meta-stable state. This is elucidated by looking at two-time quantities—mainly correlation and response functions—that depend on the two times $s$ and $t > s$ not only through their difference, even for long times. Furthermore, the decay as a function of $t$ is slower for larger $s$. This phenomenon is usually referred to as *ageing* [1]: older samples respond more slowly. The time $s$ is called the 'age' of the system or also *waiting time*, being the time elapsed since the preparation of the system. Useful quantities such as the temperature of the system cannot be defined in this genuine non-equilibrium regime.

Ageing behaviour has been experimentally observed in a variety of glassy systems (for a review see [2]), and it is expected whenever the equilibration time exceeds any experimental observation time window. This is the case for magnetic systems quenched from the high-temperature phase to their critical point or below it, when the critical relaxation and the phase ordering take place, respectively. In the latter case, both theory and experiment clearly display ageing behaviour as reviewed in [3].

An important aspect of ageing systems is that the equilibrium fluctuation–dissipation theorem (FDT) does not hold. Such 'violations' of the FDT have been the starting point for the introduction of fruitful ideas in the field of glassy systems such as non-equilibrium fluctuation–dissipation relations [4] and effective temperatures [5, 6], reviewed in [7–9]. These intriguing theoretical ideas are currently under experimental investigation in glassy [10–14] and granular [15] materials.

However, as pointed out by Cugliandolo, Kurchan and Parisi [16], 'violations' of the FDT are not peculiar to glassy and disordered systems: even a magnetic material quenched to its critical point displays non-trivial fluctuation–dissipation (FD) relations. Therefore, many aspects of ageing behaviour in classical ferromagnetic models have been investigated. Special attention has been devoted to the determination of the so-called fluctuation–dissipation ratio (FDR), in particular after the observation [17, 18] that it is universal in the sense of renormalization-group (RG) theory. Due to universality in critical phenomena, this quantity, as well as other aspects of ageing dynamics, is expected to be the same for all the systems belonging to the same universality class, irrespective of their specific realization and microscopic details. In particular, the predictions for universal quantities associated with the dynamics of real systems and of the corresponding lattice models can be obtained by means of suitable 'mesoscopic' field-theoretical models. Understanding ageing behaviour in such simple instances might also provide insight into the phenomenology of glassy systems.

The aim of this review is to give a comprehensive review of the rather large amount of theoretical ideas developed to describe ageing phenomena at the critical point, since currently many results are scattered in the literature. Particular emphasis is given to the perturbative field-theoretical approach as applied in [19–23] and to the comparison among the results of this analysis, the numerical estimates and the exact solutions that have been provided for the simplest models. In this review, we do not discuss the vast field of ageing in glassy systems, being well beyond our aims. When required, we refer the reader to the comprehensive and updated reviews available [7, 8].

The review is organized as follows.

In section 2, we recall some basic facts concerning the description of dynamics in statistical systems. Then, after a simple derivation of the FDT in equilibrium, we introduce the FDR in various forms and we remark on its importance in understanding the dynamics of slow-relaxing systems, and in particular its connection with the idea of effective temperature.

In section 3, we recall the basic steps which allow a path-integral representation of the mesoscopic dynamics, given by a suitable Langevin equation. After introducing the concept of dynamic universality classes, we briefly summarize the renormalization procedure for the corresponding field theories. The scaling forms of the response and correlation functions of the order parameter and of other observables are obtained from RG equations and short-distance expansion, following the seminal work by Janssen *et al* [24, 25].

In section 4 we consider the purely dissipative dynamics (model A in the notion of [26]) of the Landau–Ginzburg model with $O(N)$ symmetry. We review the numerical and exact results available for the relevant observables. We then consider in more detail the model within the Gaussian approximation, where the FDRs relative to different observables are equal. Finally, we review the calculation of two-loop response and correlation functions of the order parameter and observables that are quadratic in the order parameter. From the associated FDRs, it has been shown that the Gaussian equivalence of all FDRs does not hold when interactions are taken into account.

In section 5, we investigate the effects of a spatial surface on the ageing properties of model A dynamics. We report the scaling forms for correlation and response functions and we provide a solution of the model in the Gaussian approximation.

In section 6, the dynamics of a $O(N)$-symmetric conserved order parameter (model B) is analysed. We review numerical and analytical results. We derive the scaling forms for correlation and response functions solving the corresponding RG equations. These findings agree with the solutions of particular models and with some recent conjectures. We solve the model exactly in the Gaussian approximation and in the limit of an infinite number of

components of the order parameter. Beyond the Gaussian approximation we show that the FDR is not affected by one-loop corrections.

In section 7, the case of a non-conserved order parameter coupled to a conserved density (model C) is considered. The results for two-point response and correlation functions are provided in a loop expansion up to one loop.

In section 8, the purely dissipative dynamics (model A) of a weakly dilute Ising model is considered up to the first order in the loop expansion.

In section 9, the purely dissipative dynamics (model A) of a $\varphi^3$ Landau–Ginzburg Hamiltonian is considered up to the first order in $\epsilon = 6 - d$ expansion.

Finally, in section 10, we conclude the review presenting some issues that deserve further investigation.

## 2. Dynamics, the fluctuation–dissipation theorem and fluctuation–dissipation ratios

### *2.1. Dynamics*

Let us consider a system in contact with a thermal bath at a given temperature $T$. In principle the dynamics of the system is specified by its *microscopic* Hamiltonian, either classical or quantum, via the evolution equations for the density matrix and phase-space density, respectively. However, this fully microscopic approach is rarely viable for real statistical systems.

An alternative approach makes use of the representation of the dynamics as a stochastic process defined in the space of configurations of the system [27, 28]. At this description level, which can still be termed microscopic, one has to assign the transition rates between different configurations on the basis of physical considerations. Then the *master equation* rules the time evolution of the probability of finding the system in a given state. Denoting by $\mathfrak{C}$ the generic configuration of the system and by $W[\mathfrak{C} \mapsto \mathfrak{C}']$ the rate of the transition $\mathfrak{C} \mapsto \mathfrak{C}'$, the master equation reads

$$\partial_t P[\mathfrak{C}, t; \mathfrak{C}_0] = \sum_{\mathfrak{C}'} \{W[\mathfrak{C}' \mapsto \mathfrak{C}]P[\mathfrak{C}', t; \mathfrak{C}_0] - W[\mathfrak{C} \mapsto \mathfrak{C}']P[\mathfrak{C}, t; \mathfrak{C}_0]\}, \quad (1)$$

where $P[\mathfrak{C}, t; \mathfrak{C}_0]$ is the probability of finding the system in the configuration $\mathfrak{C}$ at time $t$ starting from the configuration $\mathfrak{C}_0$ at time $t = 0$. It is possible to show that an equilibrium stationary distribution function $P_{\rm eq}[\mathfrak{C}]$ (e.g., the Boltzmann–Gibbs or the microcanonical one) is reached for $t \to \infty$ if the so-called *detailed balance*

$$W[\mathfrak{C} \mapsto \mathfrak{C}']P_{\rm eq}[\mathfrak{C}] = W[\mathfrak{C}' \mapsto \mathfrak{C}]P_{\rm eq}[\mathfrak{C}'], \quad (2)$$

holds (a proof of this statement can be found in [27]). Equation (2) has to be fulfilled by the transition rates in the master equation in order to describe a dynamics towards an equilibrium state. All the dynamical properties of the system can therefore be computed once $P[\mathfrak{C}, t; \mathfrak{C}_0]$ has been determined by solving the master equation with the assigned transition rates.

Practically, solving the master equation (1) is rarely feasible. A description of the dynamics in terms of *mesoscopic* variables is in some cases preferable, since it focuses directly on those quantities that are expected to determine the dynamical properties at length and time scales (referred to as *mesoscopic*) that are much larger than the microscopic ones (at atomic or molecular level) but still small compared to the macroscopic one set by the dimension of the sample. Mesoscopic variables (or observables), such as the local magnetization density in magnetic systems or the local particle density in fluids, are obtained by averaging the corresponding microscopic quantities on mesoscopic length and time scales. This average is usually referred to as coarse graining.

A viable approach to dynamics, which was first successfully applied to the Brownian motion, consists of a description taking advantage of the separation between the typical time scale of fast (microscopic) and slow (mesoscopic) dynamical processes, clearly emerging in some cases. It is natural to assume that the dynamics of the mesoscopic observables can be described as the result of an effective slow deterministic drift towards a stationary state (equilibrium or not) and of a stochastic force that sums up the effect of the fast microscopic fluctuations. Of course, this description fails to reproduce the dynamics taking place at microscopic time and length scales. Hence, when the macroscopic physics is crucially related to some microscopic events (as claimed, for instance, in the case of rare intermittent events in some glasses [29]), such a mesoscopic description is not expected to capture the relevant physical mechanisms. We will strictly consider systems for which the mesoscopic description is feasible.

Let us assume that the mesoscopic properties of the system are described by a set of variables $\phi_i$, labelled by an index $i$. In terms of $\{\phi_i\}$, the Hamiltonian is given by $\mathcal{H}[\phi]$. The previous heuristic considerations motivate the assumption that the dynamics of the system is described by the Langevin equation

$$\partial_t \phi_i(t) = -\sum_j \mathbf{D}_{ij} \frac{\delta \mathcal{H}[\phi]}{\delta \phi_j(t)} + \zeta_i(t). \tag{3}$$

The first term in the rhs represents the deterministic evolution and $\mathbf{D}_{ij}$ is called kinetic coefficient. The stochastic nature of the equation is embodied in the random noise $\zeta_i(t)$. It is specified by its functional probability distribution function, or equivalently by the corresponding moments. In the spirit of the central limit theorem, the first two moments are sufficient to characterize the stochastic process, which hence is assumed to be Gaussian. Denoting with $\langle\cdot\rangle$ the mean over the possible realizations of the noise, we assume $\langle\zeta_i(t)\rangle = 0$ and $\langle\zeta_i(t)\zeta_j(t')\rangle = 2\mathbf{N}_{ij}\delta(t-t')$, i.e., a so-called Gaussian white noise.

From the Langevin equation (3), it is possible to derive a Fokker–Planck partial differential equation for the probability distribution function $P[\{\phi_i\}, t]$ of $\phi_i$ at time $t$, given an initial condition [30, 31, 28]

$$\partial_t P[\{\phi_i\}, t] = \frac{\delta}{\delta \phi_j}\left[\mathbf{N}_{jk}\frac{\delta P}{\delta \phi_k} + \mathbf{D}_{jk}\frac{\delta \mathcal{H}}{\delta \phi_k} P\right]. \tag{4}$$

The stationary distribution $\partial_t P[\{\phi_i\}, t] = 0$ is the Boltzmann–Gibbs probability of equilibrium statistical mechanics $P_{\mathrm{BG}}[\{\phi_i\}] \propto \mathrm{e}^{-\beta\mathcal{H}[\phi]}$ if and only if

$$\mathbf{D} = \beta\mathbf{N}. \tag{5}$$

This condition is the well-known Einstein's relation. If, given two arbitrary configurations $\{\phi_i\}$ and $\{\phi'_i\}$, a realization of the noise $\zeta_i$ exists such that starting from $\{\phi_i\}$ and evolving according to equation (3), the configuration $\{\phi'_i\}$ is reached in a finite time, the system is said to be ergodic. If ergodicity holds the long-time limit of $P[\{\phi_i\}, t]$ is $P_{\mathrm{BG}}[\{\phi_i\}]$ independently of the initial condition.

The dynamics of the system can be studied by looking at time-dependent quantities defined in terms of $\{\phi_i\}$ and averaged over the possible realizations of the noise. The simplest quantity one might consider among those which are measurable by looking at the system only at a given time $t$ (one-time quantities) is $\langle\phi_i(t)\rangle$. In the long-time limit, one expects such a quantity to reach an asymptotic value and therefore it is no longer possible to get information about the dynamics of the system. In this sense, one-time quantities are not usually enough to characterize it. The next step is to consider quantities that can be measured by looking at the configuration of the system at two different times (two-time quantities). At variance with one-time quantities, they also provide information about the dynamics of the system in the

long-time limit. Interesting two-time quantities are the correlation function of the mesoscopic variables $\{\phi_i\}$ and the response function. The former is defined as $C_{ij}(t,s) \equiv \langle \phi_i(t)\phi_j(s)\rangle$ and is related to the relaxation of spontaneous (thermal) fluctuations of the variables $\{\phi_i\}$ whereas the latter expresses the response of the system to an external perturbation. In particular, let us assume that this perturbation is due to an external field $\{h_i\}$ that couples linearly to the variables $\{\phi_i\}$ in the Hamiltonian $\mathcal{H}$ (i.e., $h_i$ is the field conjugate to $\phi_i$): $\mathcal{H}_h[\phi] = \mathcal{H}[\phi] - \sum_i h_i\phi_i$. The linear response function $R_{ij}(t,s)$ is defined by

$$R_{ij}(t,s) \equiv \left.\frac{\delta\langle\phi_i(t)\rangle_h}{\delta h_j(s)}\right|_{h=0}, \tag{6}$$

where $\langle\cdot\rangle_h$ indicates the average over the stochastic dynamics generated by equation (3) with the Hamiltonian $\mathcal{H}_h[\phi]$. Due to causality, $\phi_i(t)$ does not depend on $h_j(s)$, whenever $t < s$. As a consequence $R_{ij}(t,s) \propto \theta(t-s)$, where $\theta(t) = 1$ for $t > 0$ and 0 otherwise. In the following we set $t > s$, so that $R(s,t) = 0$.

To calculate the response function, we note that the presence of the external field $h$ is equivalent to a shift in the noise $\delta\zeta_i(t) = \mathbf{D}_{ij}h_j(t)$, so that equation (6) may be written as

$$R_{ij}(t,s) = \mathbf{D}_{jk}\left\langle\frac{\delta\phi_i(t)}{\delta\zeta_k(s)}\right\rangle = \frac{1}{2}(\mathbf{D}\mathbf{N}^{-1})_{jk}\langle\phi_i(t)\zeta_k(s)\rangle = \frac{\beta}{2}\langle\phi_i(t)\zeta_j(s)\rangle, \tag{7}$$

where we have used Einstein's relation and the algebraic identity $\langle F[\zeta]\zeta_i(t)\rangle = 2\mathbf{N}_{ij}\langle\frac{\delta F[\zeta]}{\delta\zeta_j(t)}\rangle$, with $F[\zeta]$ being an arbitrary function of the noise $\zeta$.[4] A useful relation between $R_{ij}$ and $C_{ij}$ is obtained by considering the time derivative of the correlation function $(\partial_t - \partial_s)C_{ij}(t,s) = \langle\partial_t\phi_i(t)\phi_j(s)\rangle - \langle\phi_i(t)\partial_s\phi_j(s)\rangle$ as

$$(\partial_t - \partial_s)C_{ij}(t,s) = -\left\langle\mathbf{D}_{ik}\frac{\delta\mathcal{H}[\phi]}{\delta\phi_k(t)}\phi_j(s)\right\rangle + \left\langle\phi_i(t)\mathbf{D}_{jk}\frac{\delta\mathcal{H}[\phi]}{\delta\phi_k(s)}\right\rangle - \langle\phi_i(t)\zeta_j(s)\rangle, \tag{8}$$

that can be cast in the form:

$$2T\,R_{ij}(t,s) = (\partial_t - \partial_s)C_{ij}(t,s) - A_{ij}(t,s), \tag{9}$$

where we defined the asymmetry

$$A_{ij}(t,s) = \left\langle\phi_i(t)\mathbf{D}_{jk}\frac{\delta\mathcal{H}[\phi]}{\delta\phi_k(s)}\right\rangle - \left\langle\mathbf{D}_{ik}\frac{\delta\mathcal{H}[\phi]}{\delta\phi_k(t)}\phi_j(s)\right\rangle. \tag{10}$$

Equation (9) relates correlation and response functions by means of the asymmetry. A similar equation in terms of microscopic degrees of freedom evolving according to a general class of master equations can be found in [32].

### *2.2. The fluctuation–dissipation theorem*

Here we present a simple derivation of the fluctuation–dissipation theorem within the approach to dynamics we are discussing. The same result can be obtained in many different ways, with different mathematical rigour (see, e.g., [33, 30, 8, 7, 28]).

When the system equilibrates, equation (9) assumes a simpler form. In fact the time-translational invariance (TTI) implies that the correlation and response functions satisfy $C_{ij}(t,s) = C_{ij}(t-s,0)$ and $R_{ij}(t,s) = R_{ij}(t-s,0)$. Therefore $(\partial_t - \partial_s)C_{ij}(t,s) = -2\partial_s C_{ij}(t,s)$. Moreover, equilibrium is characterized by time-reversal symmetry (TRS), and so the correlation function of two observables $\mathcal{O}_1(t)$ and $\mathcal{O}_2(t)$ satisfies $\langle\mathcal{O}_1(t)\mathcal{O}_2(s)\rangle = \langle\mathcal{O}_1(s)\mathcal{O}_2(t)\rangle$, so that the asymmetry vanishes. Taking into account these observations, one

[4] This identity is easily derived taking into account that the probability distribution function $P_N[\{\zeta_i\}]$ of the noise, being Gaussian, satisfies $-2\mathbf{N}_{jk}\delta P_N/\delta\zeta_k(s) = \zeta_j(s)P_N$. Multiplying both sides of this equation by an arbitrary $F[\zeta]$ and performing the functional integration on $\{\zeta_i\}$ (by parts on the lhs) yields the result.

concludes that

$$R_{ij}(t,s) = \beta \frac{\partial C_{ij}(t,s)}{\partial s}, \tag{11}$$

which is just one formulation of the fluctuation–dissipation theorem. Note that the FDT holds also if $C_{ij}$ is replaced by the connected correlation function $C^{(c)}_{ij}(t,s) = C_{ij}(t,s) - \langle \phi_i(t)\rangle\langle \phi_j(s)\rangle$, since due to TTI $\langle \phi_i(t)\rangle$ is independent of time in equilibrium.

In the following, we are mainly concerned with the case of systems whose physical properties close to a critical point are determined by a suitable order parameter (see subsection 3.1). In this case $\phi_i$ and $\phi_j$ stand for the values of the order parameter $\phi$ at different spatial points. Using TTI, space-translational invariance and denoting by $\mathbf{x}$ the $d$-dimensional vector between the two points $i$ and $j$, equation (11) can be written as

$$R_{\mathbf{x}}(t-s) = \beta \partial_s C_{\mathbf{x}}(t-s). \tag{12}$$

Consider now a local observable $\mathcal{O}(\mathbf{x},t)$ and define its correlation function (always with $t > s$) as $C^{\mathcal{O}}_{\mathbf{x}-\mathbf{x}'}(t,s) = \langle \mathcal{O}(\mathbf{x},t)\mathcal{O}(\mathbf{x}',s)\rangle$ and its response function

$$R^{\mathcal{O}}_{\mathbf{x}-\mathbf{x}'}(t,s) \equiv \left.\frac{\delta\langle \mathcal{O}(\mathbf{x},t)\rangle_{h_{\mathcal{O}}}}{\delta h_{\mathcal{O}}(\mathbf{x}',s)}\right|_{h_{\mathcal{O}}=0}, \tag{13}$$

where $h_{\mathcal{O}}(\mathbf{x}',s)$ is the field conjugated to $\mathcal{O}(\mathbf{x}',s)$. Equation (12) can be generalized to $\mathcal{O}$ in the form [30, 33]

$$R^{\mathcal{O}}_{\mathbf{x}}(t-s) = \beta \partial_s C^{\mathcal{O}}_{\mathbf{x}}(t-s), \tag{14}$$

i.e., the FDT holds for any observable of the system. This remarkable property allows one to define the temperature $T = \beta^{-1}$ through equation (14), independently of the specific observable used.

Whenever the system does not reach thermal equilibrium, one cannot expect that the relations in equations (11), (12) and (14) hold. In this sense their validity may be considered as a signal of the fact that the system is in equilibrium. Let us consider the following experiment. Prepare a system—a glass, a ferromagnet, etc—in an equilibrium state at a high temperature $T_0$, greater than the critical or glass transition temperature. At time $t_0 = 0$ quench the system to some temperature $T < T_0$ by taking it into contact with a thermal bath at temperature $T$ and let it evolve. On a general basis, one expects that the relaxation towards the equilibrium state corresponding to $T$ is characterized by two different regimes: (A) a transient one with non-equilibrium evolution, for $t < t_{\rm eq}(T)$, and (B) a stationary regime with equilibrium evolution for $t \gg t_{\rm eq}(T)$, where $t_{\rm eq}(T)$ is a characteristic equilibration time of the system. During (A) one expects a dependence of the behaviour of the system on initial conditions and both TRS and TTI are broken, whereas in (B) TTI and TRS are recovered: the dynamics of fluctuations is given by the 'equilibrium' one, for which FDT holds. In many systems regime (B) is not reached during experimental or even geological times because $t_{\rm eq} = \infty$ for all practical purposes. These systems always evolve out of equilibrium, even if no perturbation is acting on them. In this case the standard concepts of equilibrium statistical mechanics do not apply and in particular both TTI and TRS are broken. As a consequence two-time quantities, such as the response and correlation functions, depend separately on $s$ and $t$, even for large times.

### *2.3. The fluctuation–dissipation ratio*

A measure of the distance from equilibrium of an ageing system evolving in contact with a thermal bath at fixed temperature $T$ is the FDR [4]

$$X_{\mathbf{x}}(t,s) = \frac{T R_{\mathbf{x}}(t,s)}{\partial_s C_{\mathbf{x}}(t,s)}, \tag{15}$$

where we assume $t > s$. When the age $s$ is greater than $t_{\rm eq}(T)$, TRS and TTI hold, and the FDT yields $X_{\mathbf{x}}(t, s) = 1$. This is not generically true in the ageing regime.

The asymptotic value of the FDR

$$X^{\infty} = \lim_{s\to\infty} \lim_{t\to\infty} X_{\mathbf{x}=0}(t, s) \tag{16}$$

turns out to be a very useful quantity in the description of systems with slow dynamics, since $X^{\infty} = 1$ whenever $t_{\rm eq}(T) < \infty$, i.e., when the ageing evolution is interrupted and the system crosses over to equilibrium dynamics. Accordingly, $X^{\infty} \neq 1$ is a signal of an asymptotic non-equilibrium dynamics. Moreover $X^{\infty}$ can be used to define an effective temperature $T_{\rm eff} = T/X^{\infty}$, which might have some features of the temperature of an equilibrium system, e.g., controlling the direction of heat flows and acting as a criterion for thermalization [5]. This definition of $T_{\rm eff}$ is closely related to that introduced in the context of weak turbulence [34]. Under some assumptions it has been shown that $X_{\mathbf{x}}(t, s)$ establishes a bridge between the dynamically inaccessible equilibrium state and the asymptotic dynamics for large times [35].

In equation (12), the FDR has been defined considering the two-time correlation function $C_{\mathbf{x}}(t, s)$ and the two-time response function $R_{\mathbf{x}}(t, s)$ of the order parameter. Nevertheless it is possible to define a FDR for any observable $\mathcal{O}$:

$$X^{\mathcal{O}}_{\mathbf{x}}(t, s) = \frac{T R^{\mathcal{O}}_{\mathbf{x}}(t, s)}{\partial_s C^{\mathcal{O}}_{\mathbf{x}}(t, s)}, \qquad \text{and} \qquad X^{\infty}_{\mathcal{O}} = \lim_{s\to\infty} \lim_{t\to\infty} X^{\mathcal{O}}_{\mathbf{x}=\mathbf{0}}(t, s). \tag{17}$$

In [5] it has been shown that a thermometer coupled to the observable $\mathcal{O}$ measures, on a proper time scale, the temperature $T^{\mathcal{O}}_{\rm eff} = T/X^{\infty}_{\mathcal{O}}$, although this point has been a source of some controversy, since the measured temperature seems to depend on the thermometer employed [36]. It has been pointed out (see, e.g., [37]) that the effective temperature can be of interest in order to devise some thermodynamics for the system if its value is independent of $\mathcal{O}$, i.e., $T^{\mathcal{O}}_{\rm eff} = T_{\rm eff}$ for all $\mathcal{O}$. This has been explicitly verified for infinite-range (mean-field) glass models [5]. Beyond these cases, the observable dependence of $T^{\mathcal{O}}_{\rm eff}$ has been investigated analytically in the case of the trap model [38] and numerically for supercooled liquids [39] and in driven systems near a jamming transition [40]. The case of ageing in critical systems will be reviewed in section 4.

As already mentioned, these intriguing features are not only typical of glassy and disordered materials, but also can be generally found to some extent whenever $t_{\rm eq} = \infty$, i.e., in slow-relaxing systems such as ferromagnetic models in the low-temperature phase or at the critical point, whose dynamics is characterized by phase ordering and critical relaxation, respectively. The phase-ordering dynamics has been mainly investigated by means of analytical solutions of exactly solvable models [16, 41–45] and by Monte Carlo simulations of more realistic systems [46–50]. Due to these investigations the phenomenology of phase ordering is quite well understood [3], although some open questions remain about the scaling of the response function [51–56]. This issue goes beyond the scope of this review, therefore the interested reader is referred to the literature. During phase-ordering dynamics, the result $X^{\infty} = 0$ (corresponding to an infinite effective temperature, as experimentally found in some colloidal glasses [12]) has been put on firmer ground (see, e.g., [18]).

Particularly interesting is the case of a quench to the critical point. In fact it is possible to get insight into this problem by means of the powerful tools of renormalization group [30] and scale invariance [57] that have been developed during the last decades to investigate mainly equilibrium situations, but that can be applied to non-equilibrium relaxation as well. In particular, within this framework, it has been argued that $X^{\infty}$ and $X^{\infty}_{\mathcal{O}}$ are novel universal quantities of non-equilibrium critical dynamics [17–19, 23]. This clearly emerges from the discussion in sections 3.5.2 and 3.8.

### 2.4. The fluctuation–dissipation ratio in momentum space

In the previous section, we recalled the definition of the FDR for the response and correlation function of a space-dependent observable considered in two points a distance $\mathbf{x}$ apart, and its long-time limit $X^\infty$ for $\mathbf{x} = \mathbf{0}$. This is the original form in which the FDR has been introduced in the literature on glassy systems [4], mainly with the aim of investigating the *spatial* development of correlations.

However, within the field-theoretical approach to critical dynamics, it is more natural and computationally simpler to focus on the behaviour of observables in *momentum* space. Accordingly hereafter we mainly consider the momentum-dependent response $R_\mathbf{q}(t, s)$ and correlation $C_\mathbf{q}(t, s)$ functions, defined as the Fourier transform of the corresponding ones in real space either on the lattice or in the continuum.

To work in momentum space it is worth introducing a quantity that, just like $X_\mathbf{x}(t, s)$, 'gauges' the distance from equilibrium evolution. Its natural definition is [19]

$$\mathcal{X}_\mathbf{q}(t, s) = \frac{T R_\mathbf{q}(t, s)}{\partial_s C_\mathbf{q}(t, s)}. \tag{18}$$

Note that $\mathcal{X}_\mathbf{q}(t, s)$ is *not* the Fourier transform of $X_\mathbf{x}(t, s)$. The long-time limit

$$\mathcal{X}^\infty_{\mathbf{q}=0} \equiv \lim_{s\to\infty} \lim_{t\to\infty} \mathcal{X}_{\mathbf{q}=0}(t, s) \tag{19}$$

defines, in analogy to $X^\infty_{\mathbf{x}=\mathbf{0}}$, an effective temperature as well. It has been argued that these 'two' effective temperatures in real and momentum space are equal [19]. Hence, the effective temperature may still have a sound physical meaning. Let us outline this argument. $X_{\mathbf{x}=\mathbf{0}}$ can be written in terms of quantities in momentum space as

$$X^{-1}_{\mathbf{x}=0} \equiv \frac{\int (\mathrm{d}q)\partial_s C_\mathbf{q}(t, s)}{T \int (\mathrm{d}q) R_\mathbf{q}(t, s)} = \frac{\int (\mathrm{d}q) R_\mathbf{q}(t, s) \frac{\partial_s C_\mathbf{q}(t,s)}{T R_\mathbf{q}(t,s)}}{\int (\mathrm{d}q) R_\mathbf{q}(t, s)} = \left\langle \mathcal{X}^{-1}_\mathbf{q} \right\rangle_{R_\mathbf{q}}, \tag{20}$$

where $(\mathrm{d}q) = \mathrm{d}^d q/(2\pi)^d$. Thus $X^{-1}_{\mathbf{x}=0}$ is the average of $\mathcal{X}^{-1}_\mathbf{q}$, with weight $R_\mathbf{q}(t, s)$. In the long-time limit $t \gg s \to \infty$, the main contribution to the integral comes from the small-$\mathbf{q}$ region, provided that $R_\mathbf{q}(t, s)$ is peaked around $\mathbf{q} = 0$ with a variance that vanishes for $t \gg s \to \infty$. This is the case for most of the critical systems we are interested in. As a consequence

$$X^\infty_{\mathbf{x}=\mathbf{0}} = \mathcal{X}^\infty_{\mathbf{q}=\mathbf{0}} \equiv X^\infty. \tag{21}$$

To our knowledge, this argument is currently the only general one available to support equation (21). Several results for specific models [58–61] confirm equation (21). As it stands, the argument is expected to be valid for systems at criticality, whereas the extension to other cases might not be straightforward, being fundamental to know the form of $R_\mathbf{q}(t, s)$ for $t \gg s \to \infty$ (see also the discussion at the end of section 8). Given the importance of equation (21) for the existence of a proper, single-valued and well-defined effective temperature, a more rigorous proof and proper extensions of equation (21) would be very welcome.

We point out that definition (18) provides an alternative way to determine $X^\infty$ in numerical simulations, which is expected to increase the accuracy of the results, as shown in [58]. The advantage of using $\mathcal{X}_{\mathbf{q}=\mathbf{0}}$ is twofold. Firstly, $\mathcal{X}_{\mathbf{q}=\mathbf{0}}$ is defined by using quantities with $\mathbf{q} = \mathbf{0}$ (referred to as *coherent* observables in [58]) which are the sums over the whole sample of the corresponding quantities in real space. For instance, when considering the magnetization of a system of volume $V$, $X^\infty_{\mathbf{x}=\mathbf{0}}$ and $\mathcal{X}_{\mathbf{q}=\mathbf{0}}$ are determined by measuring the correlation of the local magnetization $m_\mathbf{x}$ and of the total magnetization $M = V^{-1} \sum_{\mathbf{x}\in V} m_\mathbf{x}$, respectively. Even if in the latter case, one has to measure connected correlation functions (which are differences

and hence more noisy than the single terms), one expects that $\mathcal{X}_{\mathbf{q}=\mathbf{0}}$ (thus $\mathcal{X}^{\infty}_{\mathbf{q}=\mathbf{0}}$) has smaller statistical fluctuations than $X_{\mathbf{x}=\mathbf{0}}$ (thus $X^{\infty}_{\mathbf{x}=\mathbf{0}}$). Secondly coherent observables directly focus on the non-equilibrated modes of the system and their FD relations typically display very early the crossover to $X^{\infty}$, when the correlation function is still comparable to its initial value. In local observables, in contrast, the various relaxing modes are mixed in, and so $X^{\infty}$ can only be measured at very large times and small ratio $s/t$, corresponding to quite small values of the correlation function. We will show explicitly these differences when discussing the Gaussian model in section 4.2.

### 2.5. *Quasi-equilibrium regime of* $X_{\mathbf{x}}(t, s)$

Throughout the review, we will be mainly interested in the long-time limit of $X_{\mathbf{x}=0}(t, s)$, i.e., in $X^{\infty}$. Nevertheless the function $X_{\mathbf{x}=0}(t, s)$ has also other interesting features in different time regimes.

In particular in the short-time regime, defined as $\delta t \equiv t - s \ll s \to \infty$, general arguments lead to the conclusion that the system is quasi-equilibrated and $\lim_{s\to\infty} X_{\mathbf{x}=0}(s^{+}, s) = 1$. This fact can be heuristically accounted for as follows. Let us consider the typical quench experiment in which the system is initially in an uncorrelated state and then, at $t = 0$, it is suddenly quenched in the low-temperature phase [3]. Thus the domain coarsening takes place. One expects that, at a given time, local quantities in two points that are inside the same domain display correlations typical of the thermodynamic phase of that domain. Deviations are expected when the two points belong to distinct domains. In the latter case the non-equilibrium behaviour is displayed. Being the short-time evolution ($\delta t \ll s$) related to processes taking place at short spatial distances (at least for systems with short-range interactions), it is equilibrium-like, resulting in $X_{\mathbf{x}=0}(s + \delta t, s) = 1$. Conversely, after a time $\delta t$ comparable with $s$, the behaviour at larger spatial distances starts influencing the dynamics and the system falls out of equilibrium.

When the system is quenched at the critical point no domains are forming. Nevertheless, at a given time, the correlation and response functions are equilibrium-like when referred to points within a given (growing) range, called dynamic correlation length, whereas they depart from equilibrium at larger distances. Thus, even in this case, $X_{\mathbf{x}=0}(t, s)$ exhibits a crossover from 1 at $t \simeq s$ to $X^{\infty}$ for $\delta t \gg s$.

The previous heuristic description can be made more rigorous considering specific models. So far quasi-equilibrium behaviour has been observed in essentially all the studied cases (see the reviews [7, 8]).

A general theorem [62], obtainable under quite reasonable assumptions for system governed by a dissipative Langevin dynamics, provides a bound for the quantity $V(t, s) = [1 - X_{\mathbf{x}=0}(t, s)]\partial_s C_{\mathbf{x}=0}(t, s)$. This bound implies that $V(t, s) \to 0$ for fixed and finite $t - s$ and $s \to \infty$. Accordingly quasi-equilibrium is reached whenever $\partial_s C_{\mathbf{x}=0}(t, s)$ does not vanish. However, for the one-dimensional Ising model, an observable $\mathcal{O}$ has been found such that $X^{\mathcal{O}}_{\mathbf{x}=0}(s, s) = 3/4$ for $s \to \infty$ [63], while $\partial_s C^{\mathcal{O}}_{\mathbf{x}=0}(t, s)$ vanishes in the same limit. This puts forward the idea that quasi-equilibrium can be defined only on a certain class of observables, termed *neutral* in [63].

Nevertheless, the notion of quasi-equilibrium is still very important: it was used in [64] to define a nominal thermodynamic temperature for models where the dynamics does not satisfy detailed balance and hence $T_{\text{bath}}$, the temperature of the thermal bath, does not exist *a priori*. In any case, this cannot be done in general non-equilibrium systems that do not satisfy detailed balance. In fact there are several examples of models where, due to a different scaling in time of the response and of the correlation functions, a non-trivial short-time $X_{\mathbf{x}=0}(t, s)$ cannot

be defined. This is the case of the contact processes [65] and of the deterministic nonlinear evolution (i.e., equation (33) with $\zeta = 0$) [66].

Let us remark that the quasi-equilibrium regime is not expected to be detectable via $\mathcal{X}_{\mathbf{q}=0}(t, s)$. Indeed, the two-point correlation and response functions involved in its definition get non-equilibrium contributions (through the Fourier transform) from the corresponding functions in real space taken at points whose distance is bigger than the domain size or than the dynamic correlation length.

## 3. Field-theoretical approach to non-equilibrium dynamics

In this section, we briefly review the renormalization-group approach to the dynamics following a quench from a high-temperature state to the critical point. In particular, we take advantage of various well-known methods to determine the scaling forms of the non-equilibrium response and correlation functions at criticality, computing the associated universal quantities in perturbation theory.

### *3.1. Critical phenomena: a brief reminder*

The macroscopic properties of statistical systems (e.g., fluids, magnetic materials, etc) are generally expected to depend strongly on the usually quite complex interactions among their microscopic constituents.

Theoretical insight can be gained only by studying analytically or numerically rather simplified models of real systems and the predicted behaviour depends on the parameters of the specific model considered. Accordingly, it is important to have a clear understanding of the connection between the model parameters and the microscopic properties of the actual system in order to provide predictions comparable with experiments. Due to this difficulty only in few cases can a quantitative comparison be made. Nevertheless, there are circumstances in which a collective behaviour emerges which is largely independent of the microscopic details of the actual system and, as a consequence, also of the particular model used to describe it. This property is known as *universality* and it naturally characterizes the physical behaviour upon approaching a critical point, where the system undergoes a continuous phase transition.

The onset of a collective behaviour is clearly revealed by the *correlation length* $\xi$, defined as the typical distance over which the microscopic variables are correlated. Far away from a critical point $\xi$ is typically of the order of the range of microscopic interactions, whereas it diverges upon approaching the critical point. Accordingly, close enough to the transition point, $\xi$ becomes mesoscopic and indeed it provides the *only* relevant length scale of a critical system.

In view of the universality of critical properties, which is justified within the renormalization-group theory (see subsection 3.5.1) and supported by experimental data [67], it is possible to study this collective behaviour in terms of suitable mesoscopic field-theoretical models, in a formal development of the Landau approach to phase transitions [68]. Indeed, as long as one is interested in the behaviour at mesoscopic length and time scales, an effective Hamiltonian which reflects the internal symmetries of the underlying microscopic system can be used. Such Hamiltonian depends only on the *order parameter* and potentially on a few other slow modes, whose actual nature is determined specifically by the system. For instance the order parameter can be identified with the magnetization in magnetic materials close to the Curie temperature, with the particle density in fluids etc. The previous considerations motivate the assumption that the static critical properties of a system with a $N$-component vector order

parameter $\varphi$, short-range interactions and $O(N)$ symmetry are captured for large distances by the Landau–Ginzburg–Wilson Hamiltonian

$$\mathcal{H}[\varphi] = \int \mathrm{d}^d x \left[ \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2} r_0 \varphi^2 + \frac{1}{4!} g_0 (\varphi^2)^2 \right], \tag{22}$$

where $r_0$ is a parameter that has to be tuned to the value $r_{0,\mathrm{crit}}$ in order to approach the critical point for $T = T_c$ and $g_0 > 0$ is the coupling constant of the theory. From now on we absorb the factor $\beta = 1/T$ in the Hamiltonian, so that $\mathcal{H}$ henceforth denotes the reduced Hamiltonian. Statistical averages of quantities $\mathcal{O}$ depending on the order parameter field $\varphi$ are then properly computed, in the equilibrium canonical ensemble, as

$$\langle \mathcal{O}[\varphi] \rangle = \frac{\int [\mathrm{d}\varphi] \mathcal{O}[\varphi]\, \mathrm{e}^{-\mathcal{H}[\varphi]}}{\int [\mathrm{d}\varphi]\, \mathrm{e}^{-\mathcal{H}[\varphi]}}. \tag{23}$$

In a first approximation (*mean field*—MF) the integral over the fields is determined by the configuration $\varphi_{\mathrm{MF}}$ which minimizes $\mathcal{H}$, i.e., satisfying $\delta\mathcal{H}[\varphi]/\delta\varphi|_{\varphi=\varphi_{\mathrm{MF}}} = 0$. Corrections coming from fluctuations around $\varphi_{\mathrm{MF}}$ can be accounted for by considering the successive terms of the formal expansion of $\mathcal{H}[\varphi]$ around $\varphi_{\mathrm{MF}}$. The term which is quadratic in $\varphi - \varphi_{\mathrm{MF}}$ defines the so-called *Gaussian approximation* of $\mathcal{H}$. Note that the free energy $\mathfrak{F} = -\ln \int [\mathrm{d}\varphi]\, \mathrm{e}^{-\mathcal{H}[\varphi]}$ is given, within the mean-field approximation, by $\mathfrak{F}_{\mathrm{MF}} = \mathcal{H}[\varphi_{\mathrm{MF}}]$. In this sense $\mathcal{H}[\varphi]$ is sometimes referred to as Landau free-energy functional.

By means of field-theoretical techniques, it is possible to determine the non-analytic behaviour observed in various thermodynamic quantities and structure factors upon approaching the critical point. Such non-analyticities, parametrized by the standard critical exponents, some associated amplitude ratios and scaling functions turn out to be universal quantities (see, e.g., [67, 69]). The values of universal quantities and scaling functions characterize the so-called *universality class* of the model.

Upon approaching a critical point, the typical time scale of dynamics of the fluctuations around the equilibrium state diverges as $\sim \xi^z$ (*critical slowing down*), where $z$ is the dynamic critical exponent. This provides the natural separation between the relevant slow evolution due to the developing collective behaviour and the fast one related to microscopic processes. This separation makes the mesoscopic description of the dynamics a particularly viable approach to the problem, as explained in section 2.1. Indeed, it allows one to compute systematically the non-analytic behaviours observed in dynamical quantities, e.g., in the low-frequency limit of the dynamic structure factor. In turn the associated universal quantities define the *dynamic universality class*. One finds that each static universality class consists of several dynamic sub-universality classes which differ, e.g., by different conserved quantities, but nonetheless exhibit the same static universal properties. In subsection 3.4, we provide some examples of physical systems belonging to the same static universality class but to different dynamical ones.

### *3.2. Non-equilibrium critical dynamics*

In the following sections, we will focus on a particular instance of non-equilibrium behaviour: the one due to a sudden thermal quench to the critical point. From the theoretical point of view this behaviour is induced by the initial conditions of the dynamics and is not generic. Indeed, as soon as the quench is done at a temperature slightly above the critical one the system thermalizes in a finite time $t_{\mathrm{eq}} \sim \xi^z$ and reaches an equilibrium state characterized by the canonical distribution function proper to the mesoscopic Hamiltonian $\mathcal{H}$ of the system. At the critical point the effects of the initial conditions persist forever and give rise to a non-equilibrium critical behaviour with some universal features.

It is possible to study theoretically these cases by simply accounting for the initial conditions in the mesoscopic evolution equation usually used to describe the dynamics of fluctuations around the equilibrium state. In particular, we consider only the case of a disordered (high-temperature) initial condition. The field-theoretical approach to this problem was developed in a seminal paper by Janssen, Schaub and Schmittmann [24], who focus on the early stage of the relaxation process after the quench, i.e., on the short-time scaling. The scaling forms obtained there for the two-time correlation and response functions $C_{\mathbf{q}}(t, s)$ and $R_{\mathbf{q}}(t, s)$ $(t > s)$, in the limit of long times, turn out to depend on the ratio $s/t$. As a consequence, the short-time regime investigated in [24] defined by the limit $s \to 0$ with fixed $t$ is equivalent to $t \to \infty$ with fixed $s$, that is the ageing limit of equations (16), (17) and (19). Accordingly we can take advantage of the analysis presented in [24] to study ageing behaviour at the critical point.

The main results obtained from the RG analysis of this problem are the scaling forms of the response and correlation functions, in equations (91) and (92), respectively. From them one deduces that

(i) The critical exponent $z$ appearing in the equilibrium and non-equilibrium scaling forms is the same [24], although its physical meaning is different.
(ii) Only *one* new independent critical exponent $\theta$ has to be introduced to describe the non-equilibrium dynamics [24], compared to those required for the equilibrium one.
(iii) The FDR $X^{\infty}$ (see equations (15) and (16)) is a universal amplitude ratio (this conclusion was first drawn in [17] on the sole basis of scaling arguments).

In the next subsections, we assume the reader to be familiar with the field-theoretical approach to critical phenomena as explained in many introductory textbooks [28, 30, 57] and we summarize here only the most relevant points, mainly concerning dynamics.

### *3.3. Path-integral representation of dynamics*

The field-theoretical approach to dynamic critical phenomena relies on a path-integral description of stochastic processes. Here we recall the basic steps which allow the construction of the path integral associated with a given Langevin equation, as developed in [70, 71] and reviewed in [25, 28, 30, 72].

Consider the following equation for the field $\varphi$ (here $\varphi$ can be either a single field, e.g., the order parameter as in models A and B or a set of slow and conserved modes in a more complex case, e.g., $\varphi \mapsto (\varphi, \varepsilon)$ in model C—cf subsection 3.4)

$$\partial_t \varphi(\mathbf{x}, t) = \mathcal{F}[\varphi(\mathbf{x}, t)] + \zeta(\mathbf{x}, t). \tag{24}$$

In this equation $\mathcal{F}$ is a local functional of $\varphi(\mathbf{x}, t)$ and $\zeta(\mathbf{x}, t)$ is a zero-mean Gaussian white noise with correlation

$$\langle \zeta(\mathbf{x}, t)\zeta(\mathbf{x}', t')\rangle = 2\mathcal{N}\delta(t - t')\delta(\mathbf{x} - \mathbf{x}'), \tag{25}$$

where $\mathcal{N}$ can either be a constant (as it is the case in model A) or a differential operator acting on $\mathbf{x}$ (as in the case of model B). In all the models we are interested in $\mathcal{F}$ has the form

$$\mathcal{F}[\varphi(\mathbf{x}, t)] = -\mathcal{D}\frac{\delta\mathcal{H}[\varphi]}{\delta\varphi(\mathbf{x}, t)}, \tag{26}$$

where $\mathcal{H}$ is a functional playing the role of a reduced Hamiltonian and $\mathcal{D}$ is a constant or a differential operator. To have TRS and thus a convergence towards the equilibrium distribution function for the field $P[\varphi] \propto \mathrm{e}^{-\mathcal{H}[\varphi]}$, the condition $\mathcal{D} = \mathcal{N}$ has to be fulfilled, which is the analogue of equation (5).

Given an initial condition $\varphi(\mathbf{x}, t_0)$, the expectation value of a generic observable $\mathcal{O}[\varphi]$ over all possible realizations of the noise $\zeta$ can be written as

$$\langle \mathcal{O} \rangle \equiv \int [\mathrm{d}\zeta] \mathcal{O}[\varphi_\zeta] P_G[\zeta] = \int [\mathrm{d}\varphi] \mathcal{O}[\varphi] \left\{ \int [\mathrm{d}\zeta] \delta(\varphi - \varphi_\zeta) P_G[\zeta] \right\}. \tag{27}$$

$P_G[\zeta]$ is the Gaussian functional probability distribution function of the noise and $\varphi_\zeta$ is the solution of equation (24) for a given realization of the noise, with the specified $\varphi(\mathbf{x}, t_0)$ as initial condition. Taking into account that

$$\delta(\varphi - \varphi_\zeta) = \delta(\partial_t \varphi - \mathcal{F}[\varphi] - \zeta) \det \left[ \partial_t - \frac{\delta \mathcal{F}}{\delta \varphi} \right], \tag{28}$$

it is possible to express the functional $\delta$-function as an exponential by introducing a complex auxiliary field $\tilde{\varphi}$: $\delta(\psi) = \int [\mathrm{d}\tilde{\varphi}] \exp\{\int \mathrm{d}t\, \mathrm{d}^d x\, \tilde{\varphi}(\mathbf{x}, t) \psi(\mathbf{x}, t)\}$. Then the average over the Gaussian noise is straightforward and leads to

$$\langle \mathcal{O} \rangle = \int [\mathrm{d}\varphi\, \mathrm{d}\tilde{\varphi}] \mathcal{O}\, \mathrm{e}^{-S_{t_0}[\varphi, \tilde{\varphi}]}, \tag{29}$$

where

$$S_{t_0}[\varphi, \tilde{\varphi}] = \int_{t_0}^{\infty} \mathrm{d}t \int \mathrm{d}^d x \, \{\tilde{\varphi}[\partial_t \varphi - \mathcal{F}[\varphi]] - \tilde{\varphi} \mathcal{N} \tilde{\varphi}\}. \tag{30}$$

The functional $S_{t_0}[\varphi, \tilde{\varphi}]$, usually referred to as the dynamic functional, is the starting point for the field-theoretical approach to dynamics. Note that $\tilde{\varphi}(\mathbf{x}, t)$ has a clear physical meaning. Indeed, given an external field $h$ coupled linearly to $\varphi$, one has $\mathcal{H}[\varphi, h] = \mathcal{H}[\varphi] - \beta \int \mathrm{d}^d x\, h\varphi$. This implies, following equation (30), that $\tilde{\varphi}$ is conjugated in $S_{t_0}[\varphi, \tilde{\varphi}]$ to the external field $h$, i.e., $S_{t_0}[\varphi, \tilde{\varphi}, h] = S_{t_0}[\varphi, \tilde{\varphi}] - \beta \int_{t_0}^{\infty} \mathrm{d}t \int \mathrm{d}^d x\, \tilde{\varphi} \mathcal{D} h$. As a consequence, the linear response of an observable $\mathcal{O}$ to the field $h$ is given by

$$\frac{\delta \langle \mathcal{O} \rangle}{\delta h(\mathbf{x}, s)} = \beta \langle \tilde{\varphi}(\mathbf{x}, s) \mathcal{D} \mathcal{O} \rangle. \tag{31}$$

For this reason $\tilde{\varphi}(\mathbf{x}, s)$ is termed response field. In particular the response function (of the order parameter) reads

$$R_{\mathbf{x}-\mathbf{x}'}(t, s) \equiv \left. \frac{\delta \langle \varphi(\mathbf{x}, t) \rangle_h}{\delta h(\mathbf{x}', s)} \right|_{h=0} = \beta \langle \tilde{\varphi}(\mathbf{x}', s) \mathcal{D} \varphi(\mathbf{x}, t) \rangle. \tag{32}$$

From now on we absorb the factor $T = \beta^{-1}$ in the definition of the response function.

We remark that in equation (30) the term corresponding to the determinant in equation (28) is missing. To be properly evaluated, it requires a discretization of the Langevin equation and eventually its expression depends on the chosen discretization [25, 28, 73]. Nevertheless the result of the computation of averages such as equation (29) is actually independent of the particular choice [72] which can be made in such a way to render the determinant equal to 1. In turn, this implies that $R_{\mathbf{x}-\mathbf{x}'}(t, t) \propto \langle \varphi(\mathbf{x}, t) \zeta(\mathbf{x}', t) \rangle = 0$, corresponding to the so-called Îto prescription in stochastic calculus (see, e.g., [27, 74]). Accordingly, in the perturbative expansion of averages as (29), all the diagrams with at least one loop of the response function do not contribute.

### *3.4. Dynamic universality classes*

A classification of several dynamic universality classes was done in the early seventies and is reviewed in the classical paper by Hohenberg and Halperin [26]. These universality classes have been named with capital letters, from A to J (some new classes have been added to the

original classification of [26]). A rather complete set of two-loop results for the equilibrium dynamics has been only recently reached (see [75] for a brief review).

In the following, we introduce the dynamic universality classes that we will consider in this review.

*3.4.1. Purely dissipative relaxation: model A.* The purely dissipative dynamics of a $N$-component field $\varphi_i$ $(i = 1, \ldots, N)$ can be specified in terms of the stochastic Langevin equation

$$\partial_t \varphi_i(\mathbf{x}, t) = -\Omega \frac{\delta \mathcal{H}[\varphi]}{\delta \varphi_i(\mathbf{x}, t)} + \zeta_i(\mathbf{x}, t), \tag{33}$$

where $\mathcal{H}[\varphi]$ is the static reduced Hamiltonian of the model, $\Omega$ is a kinetic coefficient and $\zeta_i(\mathbf{x}, t)$ a zero-mean Gaussian white noise with correlations

$$\langle \zeta_i(\mathbf{x}, t)\zeta_j(\mathbf{x}', t')\rangle = 2\Omega\delta(\mathbf{x} - \mathbf{x}')\delta(t - t')\delta_{ij}, \tag{34}$$

where $T$ is the bath temperature. Equation (33) is a special case of the more general one introduced in section 3.3, with $\mathcal{D} = \Omega, \mathcal{N} = \Omega$ and it fulfils Einstein's relation $\mathcal{D} = \mathcal{N}$.

The critical dynamics of some anisotropic magnets and alloys [26] are described by equations (33) and (34) with $\mathcal{H}$ given by equation (22) with $N = 1$, i.e., by the effective Hamiltonian of the Ising universality class. Model A also describes the critical dynamics of kinetic spin models on the lattice and spin-flip sampling, i.e., in which the elementary step amounts to an arbitrary change in the orientation of the spin in a given site, performed with proper rates (e.g., the well-known Glauber dynamics [76]). In the case of $N$-component spin models on a regular lattice and $O(N)$-symmetric short-range interactions, the critical dynamics is described by equation (33) where $\mathcal{H}$ is given by equation (22).

The dynamical properties of model A may be worked out by representing the Langevin equation (33) as a dynamical functional, following the method outlined in the previous section. The resulting action is

$$S[\varphi, \tilde{\varphi}] = \int_{t_0}^{\infty} \mathrm{d}t \int \mathrm{d}^d x \left[ \tilde{\varphi}\partial_t \varphi + \Omega\tilde{\varphi}\frac{\delta \mathcal{H}[\varphi]}{\delta \varphi} - \tilde{\varphi}\Omega\tilde{\varphi} \right]. \tag{35}$$

To specify completely the dynamics, one has to provide the initial condition for the field $\varphi(\mathbf{x}, t)$: $\varphi(\mathbf{x}, t_0) = \varphi_0(\mathbf{x})$. More generally one can assign a probability distribution function for the initial condition, in the form $\exp\{-\mathcal{H}_0[\varphi_0]\}$ and then average over the initial field $\varphi_0(\mathbf{x})$. If the system is already in thermal equilibrium at time $t_0$, then $\mathcal{H}_0[\varphi_0] = \mathcal{H}[\varphi_0]$ and one can equivalently extend the time integration in $S[\varphi, \tilde{\varphi}]$, from $-\infty$ to $\infty$ [24]. This is possible since, assuming ergodicity, independently of the initial condition in the far past, the same stationary order parameter distribution is reached at time $t_0$. The resulting theory is translational invariant both in space and time, and given that the Einstein's relation is fulfilled, equal-time correlation functions can be computed directly using the functional distribution $\mathrm{e}^{-\mathcal{H}[\varphi]}$.

*3.4.2. Conserved order parameter: model B.* In some cases model A is not suited to describe the dynamics of physical systems. For instance, when the order parameter is related to the density of particles in a fluid, as it is the case when studying the liquid–gas critical point, one expects the continuity equation to be obeyed. The simplest model in which such a local conservation law is implemented is known as model B, according to the classification of [26]. In particular, the dynamics of the scalar order parameter $\varphi$ is given by

$$\partial_t \varphi(\mathbf{x}, t) + \nabla \cdot \mathbf{J}(\mathbf{x}, t) = 0, \tag{36}$$

where $\mathbf{J}(\mathbf{x}, t)$ is a fluctuating driving force. Thinking of $\varphi$ as a particle density one expects the deterministic driving force to be related to the gradient of some sort of local chemical potential. Indeed it is generally assumed that

$$\mathbf{J}(\mathbf{x}, t) = -\sigma \nabla_{\mathbf{x}} \frac{\delta \mathcal{H}[\varphi]}{\delta \varphi(\mathbf{x}, t)} + \mathbf{J}_{\zeta}(\mathbf{x}, t), \tag{37}$$

where $\sigma$ is a kinetic coefficient and $\mathbf{J}_{\zeta}$ a Gaussian zero-mean random current with correlations

$$\langle J_{\zeta,i}(\mathbf{x}, t) J_{\zeta,j}(\mathbf{x}', t') \rangle = 2\sigma \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') \delta_{ij}. \tag{38}$$

In the example given above, $\sigma$ is related to the mobility whereas $\bar{\mu}(\mathbf{x}, t) = \delta \mathcal{H}[\varphi]/\delta \varphi(\mathbf{x}, t)$ represents a sort of chemical potential whose inhomogeneities drive the diffusion of the particles.

It is not difficult to recast the dynamic model specified by the previous equations in the form of the Langevin equation (33), with the noise correlation (34) (where $\zeta = -\nabla \cdot \mathbf{J}_{\zeta}$), $N = 1$ and $\Omega \to -\sigma \nabla_{\mathbf{x}}^2$ [26, 30].

Model B describes the critical dynamic properties of some uniaxial ferromagnets [26] which belong to the Ising universality class, i.e., the effective Hamiltonian $\mathcal{H}$ in equation (37) is given by equation (22) with $N = 1$. Moreover model B describes the critical behaviour of lattice spin models with spin-exchange sampling (also known as Kawasaki dynamics [77]), characterized by an elementary step which amounts to an exchange between the spins in two neighbouring sites.

The dynamical functional associated with model B is given by

$$S[\varphi, \tilde{\varphi}] = \int \mathrm{d}t \int \mathrm{d}^d x \left[ \tilde{\varphi} \partial_t \varphi - \sigma \tilde{\varphi} \nabla_{\mathbf{x}}^2 \frac{\delta \mathcal{H}[\varphi]}{\delta \varphi} + \tilde{\varphi} \sigma \nabla_{\mathbf{x}}^2 \tilde{\varphi} \right]. \tag{39}$$

The initial condition can be accounted for as described in the case of model A.

*3.4.3. Coupling of a conserved density to the non-conserved order parameter: model C.* The models of dynamics considered in the previous subsections involve only the time evolution of the order parameter $\varphi$, subjected to the thermal fluctuations due to the coupling to the thermal bath. These models, although quite rich and complex in their phenomenology, are usually too oversimplified to be able to capture the relevant features of real systems, in particular those with a vector order parameter. In many cases, the order parameter is not the only relevant slow variable that has to be taken into account. For instance, in the case of a one-component fluid close to its critical point, the conserved order parameter interacts with three slow hydrodynamic modes [31].

The simplest model with two interacting fields, called model C, consists of a non-conserved $N$-component vector order parameter $\varphi(\mathbf{x}, t)$ coupled to a non-critical conserved density $\varepsilon(\mathbf{x}, t)$.

The dynamics is specified by the following coupled stochastic Langevin equations

$$\partial_t \varphi(\mathbf{x}, t) = -\Omega \frac{\delta \mathcal{H}[\varphi, \varepsilon]}{\delta \varphi(\mathbf{x}, t)} + \zeta_{\varphi}(\mathbf{x}, t), \tag{40}$$

$$\partial_t \varepsilon(\mathbf{x}, t) = \Omega \rho \nabla_{\mathbf{x}}^2 \frac{\delta \mathcal{H}[\varphi, \varepsilon]}{\delta \varepsilon(\mathbf{x}, t)} + \zeta_{\varepsilon}(\mathbf{x}, t), \tag{41}$$

where $\Omega$ and $\rho$ are the kinetic coefficients, $\mathcal{H}[\varphi, \varepsilon]$ the reduced Hamiltonian of the system and $\zeta_{\varphi}(\mathbf{x}, t)$, $\zeta_{\varepsilon}(\mathbf{x}, t)$ zero-mean Gaussian noises with

$$\langle \zeta_{\varphi}(\mathbf{x}, t) \zeta_{\varphi}(\mathbf{x}', t') \rangle = 2\Omega \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'), \tag{42}$$

$$\langle \zeta_{\varepsilon}(\mathbf{x}, t) \zeta_{\varepsilon}(\mathbf{x}', t') \rangle = -2\rho\Omega \nabla_{\mathbf{x}}^2 \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'). \tag{43}$$

Note that Einstein's relation is fulfilled by this system of coupled Langevin equations, therefore the equilibrium distribution function of the fields $\varepsilon$ and $\varphi$ is given by $\exp\{-\mathcal{H}[\varphi, \varepsilon]\}$, suitably normalized.

In the case of a system with short-range $O(N)$-symmetric interactions, the Hamiltonian $\mathcal{H}[\varphi, \varepsilon]$ in equation (40) is given by

$$\mathcal{H}[\varphi, \varepsilon] = \mathcal{H}[\varphi] + \int \mathrm{d}^d x \left[\frac{1}{2}\varepsilon^2 + \frac{1}{2}\gamma_0 \varepsilon \varphi^2\right], \tag{44}$$

where $\mathcal{H}[\varphi]$ is the $O(N)$-symmetric Landau–Ginzburg Hamiltonian reported in equation (22) and $\gamma_0$ the coupling constant between $\varphi(\mathbf{x}, t)$ and $\varepsilon(\mathbf{x}, t)$. In particular, this coupling does not change the critical static properties of the field $\varphi$ as it can be seen by computing (through a Gaussian integration) the effective Hamiltonian $\mathcal{H}_{\mathrm{eff}}[\varphi]$ defined by $\exp\{-\mathcal{H}_{\mathrm{eff}}[\varphi]\} = \int [\mathrm{d}\varepsilon] \exp\{-\mathcal{H}[\varphi, \varepsilon]\}$. Indeed, it turns out that $\mathcal{H}_{\mathrm{eff}}[\varphi] = \mathcal{H}[\varphi]$ (see equation (22)) where $g_0 \mapsto g_0' \equiv g_0 - 12\gamma_0^2$. On the same footing, one can show that $\varepsilon$-field equilibrium correlation functions are related to $\varphi^2$-field correlation functions [30]. For this reason $\varepsilon$ is also referred to as energy density, given that $\varphi^2$ is conjugate to the temperature ($\propto r_0$) in equation (22) and therefore the derivatives of the free energy with respect to the temperature give rise to $\varphi^2(\varepsilon)$-correlation functions.

Some lattice models belonging to this universality class are discussed in [78]. Among real systems whose dynamic critical properties are described by model C with Hamiltonian (44) and $N = 1$, we mention the case of structural (displacive) transitions in crystalline materials and some phase transition in uniaxial antiferromagnets [26].

Dynamical correlation functions, generated by the Langevin equations (40) and averaged over the noises $\zeta_\varphi$ and $\zeta_\varepsilon$, may be obtained by means of the field-theoretical action

$$S = \int \mathrm{d}^d x \, \mathrm{d}t \left[\tilde{\varphi}\partial_t \varphi + \Omega\tilde{\varphi}\frac{\delta \mathcal{H}[\varphi, \varepsilon]}{\delta \varphi} - \tilde{\varphi}\Omega\tilde{\varphi} + \tilde{\varepsilon}\partial_t \varepsilon - \rho\Omega\tilde{\varepsilon}\nabla_{\mathbf{x}}^2 \frac{\delta \mathcal{H}[\varphi, \varepsilon]}{\delta \varepsilon} + \tilde{\varepsilon}\rho\Omega\nabla_{\mathbf{x}}^2\tilde{\varepsilon}\right], \tag{45}$$

where two response fields $\tilde{\varphi}(\mathbf{x}, t)$ and $\tilde{\varepsilon}(\mathbf{x}, t)$ associated with $\varphi(\mathbf{x}, t)$ and $\varepsilon(\mathbf{x}, t)$ have been introduced. It is easy to read from equation (45) and (44) the interaction vertices, given by $-\Omega g_0 \tilde{\varphi}\varphi^3/3!$ (as in the case of model A), $-\Omega\gamma_0\varepsilon\tilde{\varphi}\varphi$ and $\rho\Omega\gamma_0\varphi^2\nabla^2\tilde{\varepsilon}/2$.

To take into account the effect of the initial condition one has also to average over the possible initial configurations of both the order parameter $\varphi_0(\mathbf{x}) = \varphi(\mathbf{x}, t = 0)$ and the conserved density $\varepsilon_0(\mathbf{x}) = \varepsilon(\mathbf{x}, t = 0)$ with a probability distribution $\mathrm{e}^{-H_0[\varphi_0, \varepsilon_0]}$ given by [81]

$$H_0[\varphi_0, \varepsilon_0] = \int \mathrm{d}^d x \left\{\frac{\tau_0}{2}[\varphi_0(\mathbf{x}) - u(\mathbf{x})]^2 + \frac{1}{2c_0}[\varepsilon_0(\mathbf{x}) - v(\mathbf{x})]^2\right\}. \tag{46}$$

This specifies an initial state $u(\mathbf{x})$ for $\varphi(\mathbf{x}, t)$ and $v(\mathbf{x})$ for $\varepsilon(\mathbf{x}, t)$ with correlations proportional to $\tau_0^{-1}$ and $c_0$, respectively.

The model with a conserved order parameter coupled to a conserved density is called model D [26].

*3.4.4. Other dynamic universality classes.* The models of critical dynamics introduced so far are in many cases still too simple to describe the critical dynamic properties of a variety of actual materials.

For instance let us briefly discuss the case of isotropic ferromagnets, microscopically described by the so-called Heisenberg model for the three-component spin variables (see, e.g., [28]). In the absence of symmetry-breaking fields the model has a $O(3)$ symmetry. At a mesoscopic level the system is described by a three-component order parameter $\mathbf{S}(\mathbf{x}, t)$, representing the local magnetization, which turns out to be a dynamically conserved quantity. In view of the underlying $O(3)$ symmetry, the universal aspects of the equilibrium critical

properties of the model are correctly captured by the Hamiltonian (22) with $N = 3$ [67] ($\varphi_i \mapsto S_i$). The mesoscopic dynamics of order parameter $\mathbf{S}(\mathbf{x}, t)$ has to account both for the conservation law mentioned above and for the expected Larmor's precession of $\mathbf{S}(\mathbf{x}, t)$ in the local magnetic field $\delta\mathcal{H}[\mathbf{S}]/\delta\mathbf{S}(\mathbf{x}, t)$ generated by neighbouring spins. The Langevin equation accounting for these features and the associated universality class are known as model J [26]:

$$\partial_t \mathbf{S}(\mathbf{x}, t) = -g_L \mathbf{S}(\mathbf{x}, t) \times \frac{\delta\mathcal{H}[\mathbf{S}]}{\delta\mathbf{S}(\mathbf{x}, t)} + \sigma\nabla_{\mathbf{x}}^2 \frac{\delta\mathcal{H}[\mathbf{S}]}{\delta\mathbf{S}(\mathbf{x}, t)} + \boldsymbol{\zeta}(\mathbf{x}, t), \tag{47}$$

where $g_L$ is the coupling constant of the precession term, $\sigma$ the kinetic coefficient (analogous to that of model B) and $\boldsymbol{\zeta}$ a Gaussian random current with correlations given by equation (34) where $\Omega \mapsto -\sigma\nabla_{\mathbf{x}}^2$. Apart from the precession term (which is an example of the general class of *reversible mode couplings*, see [28, 30] for details), the dynamics of each single component $S_i$ is given by a model B and satisfies Einstein's relation. The precession term, on the other hand, does not influence the equilibrium distribution which is still given by $\exp\{-\mathcal{H}[\mathbf{S}]\}$, properly normalized [28, 30]. We do not discuss the properties characterizing this universality class but we mention only that they agree with what has been observed in experiments [26].

The critical properties of an isotropic antiferromagnet close to the Néel temperature are instead described by a non-conserved three-component order parameter $\mathbf{m}_s(\mathbf{x}, t)$ (the local staggered magnetization) which is dynamically coupled, via precession-like terms, to a three-component conserved density representing the local magnetization $\mathbf{m}(\mathbf{x}, t)$. The resulting dynamic universality class is known as model G and its predictions agree with the experimental results on actual magnetic systems. We remark that the equilibrium properties of the model are given by the Hamiltonian (22) (with $N = 3$) for the order parameter $\mathbf{m}_s$ [67]. Accordingly, models G and J provide a non-trivial example of two different dynamic universality classes of real magnetic systems that reduce to the same static universality class. Other examples are provided by the dynamic universality classes describing a planar magnet and the superfluid $^4$He, known as model E and F, respectively [26], which belong to the same static universality class as the Hamiltonian (22) with two-component order parameter (i.e., $O(2)$ symmetry).

Since we do not consider these models in what follows, we do not discuss any further details of the associated dynamic universality classes and refer the interested reader to the available review [26] and textbooks [28, 31] on the subject.

### *3.5. Renormalization and scaling of a purely dissipative critical system*

As we discussed in subsection 2.1, the quantities one is interested in when studying dynamical processes are the response and correlation functions. In the following, we will generically refer to them as correlation functions. In subsection 3.3, we described the method that allows the computation of such functions as averages of form (29) where the specific expression of the functional $S_{t_0}$ depends on the considered dynamics. Field-theoretical methods can be applied to compute such averages. As usual, when going beyond the Gaussian approximation (i.e., considering the anharmonic terms in the expansion of $S_{t_0}$), one faces the problem of ultraviolet (UV) divergences that can be regularized by standard methods and removed by means of a standard renormalization procedure. The existence of a well-defined renormalized theory allows the derivation of RG equations for the correlation functions. In particular, their solutions highlight the scaling properties of these functions in the large-distance long-time limit (the so-called infrared (IR) behaviour) which can be computed by means of the renormalized perturbation theory.

To illustrate the general procedure, we consider as a specific example the model A dynamics of a $\varphi^4$ field theory. All the following basic steps can be carried over to more complex cases.

*3.5.1. RG, $\epsilon$ expansion and minimal subtraction.* The RG idea is to construct a scale-dependent effective action $S_\varkappa[\varphi, \tilde{\varphi}]$ (with $\varkappa \geqslant 1$) which has connected correlation functions $G_{n,\tilde{n}} = \langle [\varphi]^n [\tilde{\varphi}]^{\tilde{n}} \rangle_c$ satisfying

$$G^{(\varkappa)}_{n,\tilde{n}}(\{\mathbf{x}, t\}) = \mathcal{Z}^{-n/2}(\varkappa)\tilde{\mathcal{Z}}^{-\tilde{n}/2}(\varkappa) G_{n,\tilde{n}}(\{\varkappa\mathbf{x}, \varkappa^z t\}) + \cdots, \tag{48}$$

where $G^{(\varkappa)}_{n,\tilde{n}}$ is computed with the effective action $S_\varkappa$, whereas $G_{n,\tilde{n}}$ is obtained with the original action $S$. One can arbitrarily fix $S_{\varkappa=1} \equiv S$. The ellipsis in equation (48) stands for functions decreasing faster than any power of $\varkappa$ for $\varkappa \to \infty$, $\{\mathbf{x}, t\}$ stands for the collection of the $n + \tilde{n}$ spacetime points, and we allow an anisotropic scaling between space and time through the exponent $z$. The mapping $S[\varphi, \tilde{\varphi}] \to S_\varkappa[\varphi, \tilde{\varphi}]$ is called RG transformation and $\{S_\varkappa[\varphi, \tilde{\varphi}]\}_{\varkappa \geqslant 1}$ is called the trajectory of $S[\varphi, \tilde{\varphi}]$ under the RG flow. Various RG transformations differ by the form of $\mathcal{Z}(\varkappa)$, $\tilde{\mathcal{Z}}(\varkappa)$, and of $S_\varkappa$. In explicit constructions, the transformations are generated by integrations over the large-momentum fluctuation modes of the fields.

The coupling constants appearing in $S_\varkappa[\varphi, \tilde{\varphi}]$ are explicit function of $\varkappa$. If for $\varkappa \to \infty$, the action $S_\varkappa[\varphi, \tilde{\varphi}]$ has a limit $S^*[\varphi, \tilde{\varphi}]$, called the fixed-point action, the correlation functions behave like

$$G_{n,\tilde{n}}(\{\varkappa\mathbf{x}, \varkappa^z t\}) \sim \mathcal{Z}^{n/2}_*(\varkappa)\tilde{\mathcal{Z}}^{\tilde{n}/2}_*(\varkappa) G^*_{n,\tilde{n}}(\{\mathbf{x}, t\}) \qquad \text{for} \quad \varkappa \to \infty, \tag{49}$$

where $*$ is used to denote fixed-point quantities and $G^*_{n,\tilde{n}}$ is computed with the fixed-point action $S^*$. It is possible to show that under some reasonable assumptions [30] $\mathcal{Z}_*(\varkappa) = \varkappa^{-2d_\varphi}$ and $\tilde{\mathcal{Z}}_*(\varkappa) = \varkappa^{-2d_{\tilde{\varphi}}}$. Accordingly, correlation functions have a scaling behaviour at large distances and long times:

$$G_{n,\tilde{n}}(\{\varkappa\mathbf{x}, \varkappa^z t\}) \sim \varkappa^{-nd_\varphi - nd_{\tilde{\varphi}}} G^*_{n,\tilde{n}}(\{\mathbf{x}, t\}) \qquad \text{for} \quad \varkappa \to \infty, \tag{50}$$

where the exponents $d_\varphi$, $d_{\tilde{\varphi}}$ (the so-called scaling dimensions of the fields) and $z$ are properties of the fixed point, i.e., they depend only on $S^*[\varphi, \tilde{\varphi}]$. Correlation functions of all those actions that flow under RG transformations into the same fixed-point action $S^*$ display the same critical scaling behaviour: this provides the basis for the universality observed in critical phenomena. Within this framework a given universality class is characterized by the values of the exponents $d_\varphi$, $d_{\tilde{\varphi}}$ and $z$ and by the scaling function $G^*_{n,\tilde{n}}$. There are several ways and approximations that allow us to calculate such quantities in physical dimensions $d = 2, 3$.

Renormalization-group equations for the correlation functions can be conveniently derived within the field-theoretical approach. They eventually lead to a relation of form (48) for correlation functions. The first step is to regularize the theory in such a way to render finite those Feynman diagrams of the perturbative expansion that would be otherwise UV divergent. To this end we will assume dimensional regularization [30]. The original UV divergences of the four-dimensional theory appear in the correlation functions as poles when $\epsilon = 4 - d \to 0$, i.e., when the regularization is formally removed. Let us consider the model A dynamics with Hamiltonian given by equation (22). In this specific case one introduces the *renormalized parameters* as

$$\begin{aligned} &\overset{\circ}{\varphi} = Z^{1/2}\varphi, \qquad &&\overset{\circ}{\tilde{\varphi}} = \tilde{Z}^{1/2}\tilde{\varphi}, \\ &\overset{\circ}{\Omega} = Z_\omega \Omega, \qquad &&r_0 \equiv \overset{\circ}{r} = Z^{-1} Z_r r, \qquad \text{and} \quad g_0 \equiv \overset{\circ}{g} = Z_g \mu^{4-d} g. \end{aligned} \tag{51}$$

Here the parameters with the superscript $\circ$ are those originally appearing in equation (35) (*bare parameters*) whereas the others represent the renormalized ones. Note that the renormalized

coupling constant $g$ is defined in such a way to be dimensionless. $\mu$ is an arbitrary scale with the dimension of a mass that in general enters in the renormalization of the parameters that are not dimensionless. The $Z$ factors are called renormalization constants and can be computed in many different ways. In the following we will use the so-called minimal subtraction (MS) scheme [30] which allows one to study directly the critical theory, hence we fix $r = 0$ in what follows. Within MS the renormalization constants take the form $Z = 1 + \sum \alpha_n(g)/\epsilon^n$, where the coefficient $\alpha_n(g) = O(g^n)$ is computed order by order in perturbation theory. It is possible to show that the FDT yields the relation $Z_\omega = (Z/\tilde{Z})^{1/2}$ [30]. Once the correlation functions of the fields are expressed in terms of the renormalized parameters only, they turn out to be regular functions for $\epsilon \to 0$.

To renormalize $G_{n,\tilde{n}}$ one uses the $Z$-factors in equations (51) obtaining

$$\overset{\circ}{G}_{n,\tilde{n}} = Z^{n/2}\tilde{Z}^{\tilde{n}/2}G_{n,\tilde{n}}, \tag{52}$$

where the rhs is expressed in terms of renormalized quantities. The RG equations may be derived by exploiting the fact that the bare correlation functions are independent of the mass scale $\mu$ introduced to define the renormalized theory

$$\mu\partial_\mu \overset{\circ}{G}_{n,\tilde{n}}|_0 = 0, \tag{53}$$

where the notation $|_0$ reminds that the derivative is taken with fixed bare parameters. As we show below, this equation defines a RG flow in the parameter space $\{g, \mu, \Omega\}$.

In terms of renormalized quantities, equation (53) reads

$$\left[\mu\frac{\partial}{\partial\mu} + \frac{\tilde{n}}{2}\tilde{\gamma} + \frac{n}{2}\gamma + \gamma_\omega\Omega\frac{\partial}{\partial\Omega} + \beta_g\frac{\partial}{\partial g}\right] G_{n,\tilde{n}}(\{\mathbf{x}, t\}; g, \mu, \Omega) = 0, \tag{54}$$

where the so-called Wilson's functions are defined by

$$\gamma(g) = \mu\partial_\mu \ln Z|_0, \qquad \tilde{\gamma}(g) = \mu\partial_\mu \ln \tilde{Z}|_0, \qquad \gamma_\omega(g) = \mu\partial_\mu \ln \Omega|_0, \tag{55}$$

and

$$\beta(g) = \mu\partial_\mu g|_0, \tag{56}$$

which are regular for $\epsilon = 0$ [30].

To solve RG equation (54), the method of characteristic is usually employed. One introduces a continuous real scale parameter $l$, in terms of which the flowing mass scale $\mu$ reads $\mu(l) = \mu l$ and $\mu$ plays the role of an initial condition for $l = 1$. The IR behaviour is obtained for $l \to 0$. Then one defines effective $l$-dependent parameters by solving the ordinary differential equations

$$l\frac{\mathrm{d}\Omega(l)}{\mathrm{d}l} = \Omega(l)\gamma_\omega(l), \qquad l\frac{\mathrm{d}g(l)}{\mathrm{d}l} = \beta(l), \quad \text{etc}, \tag{57}$$

with $\Omega(l = 1) = \Omega$, $g(l = 1) = g$ etc, where the RG functions on the rhs depend on $l$ through $g(l)$, e.g., $\gamma(l) = \gamma(g(l))$. The RG equation for the renormalized correlation functions becomes an ordinary differential equation

$$\left[\frac{\tilde{n}}{2}\tilde{\gamma}(l) + \frac{n}{2}\gamma(l) + l\frac{\mathrm{d}}{\mathrm{d}l}\right] G_{n,\tilde{n}}(\{\mathbf{x}, t\}; g(l), \mu(l), \Omega(l)) = 0, \tag{58}$$

whose solution is

$$G_{n,\tilde{n}}(\{\mathbf{x}, t\}; g(l), \mu(l), \Omega(l)) = G_{n,\tilde{n}}(\{\mathbf{x}, t\}; g, \mu, \Omega) \exp\left\{\int_l^1 \frac{\mathrm{d}l'}{2l'}[n\gamma(l') + \tilde{n}\tilde{\gamma}(l')]\right\}. \tag{59}$$

Physically, varying the parameter $l$, one changes the scale at which the model is explored. The infrared regime is reached for $l \to 0$. The flowing parameters $\Omega(l)$ and $g(l)$ can be

interpreted as effective values of the original ones on different length/time scales set by $\mu l$. They evolve under scale transformation $\mu \to \mu l$ according to the flow equations that are determined by the required renormalizations through the functions $\gamma$, $\tilde{\gamma}$, $\gamma_\omega$ and $\beta$ (see equations (55) and (56)).

Under a scale transformation, the dimensionless renormalized coupling $g$ changes according to equation (56). Therefore, $g(l)$ increases for $l \to 0$ when the $\beta$ function is negative and decreases in the opposite case. Fixed points for the flow of $g$ correspond to zeroes $\bar{g}$ of the $\beta$ function, i.e., $\beta(\bar{g}) = 0$. Those where $\beta'(\bar{g}) < 0$ are not stable in the limit $l \to 0$ (IR behaviour) we are interested in: the effective coupling moves away from them. Conversely, when $\beta'(\bar{g}) > 0$, the zero is stable. Indeed, linearizing the flow equation for $g$ close to $\bar{g}$ one finds that $g(l) - \bar{g} = (g - \bar{g})l^{\beta'(\bar{g})}$ for $l \ll 1$. Let us assume we find such an IR-stable fixed point for $g = g^*$. We denote the fixed-point values of the RG functions as $\eta = \gamma(g^*)$, $\tilde{\eta} = \tilde{\gamma}(g^*)$, $\eta_\omega = \gamma_\omega(g^*) = (\tilde{\eta} - \eta)/2$ (this last equality follows from the FDT), and $z = 2 + \eta_\omega$. Accordingly, $\Omega(l \to 0) = \Omega l^{z-2}$.

The leading IR behaviour is obtained from equation (59)

$$G_{n,\tilde{n}}(\{\mathbf{x}, t\}; g, \mu, \Omega) \simeq_{l\to 0} l^{\eta n/2 + \tilde{\eta}\tilde{n}/2} G_{n,\tilde{n}}(\{\mathbf{x}, t\}; g(l), \mu(l), \Omega(l)). \tag{60}$$

From the dimensional analysis one finds that

$$G_{n,\tilde{n}}(\{\mathbf{x}, t\}; g, \mu, \Omega) = \ell^{-n(d-2)/2 - \tilde{n}(d+2)/2} G_{n,\tilde{n}}(\{\mathbf{x}/\ell, t/\tau\}; g, \mu\ell, \Omega\tau/\ell^2), \tag{61}$$

where $\ell$ and $\tau$ are arbitrary units of measure of length and time, respectively. Applying this equation to the rhs of equation (60) with $\ell = (\mu l)^{-1}$ and $\tau = \Omega^{-1}\ell^2 l^{2-z}$, one finally arrives at the leading IR scaling behaviour of the correlation functions

$$G_{n,\tilde{n}}(\{\mathbf{x}, t\}; g, \mu, \Omega) = l^{\delta(n,\tilde{n})} \mu^{n(d-2)/2 + \tilde{n}(d+2)/2} G_{n,\tilde{n}}(\{l(\mu\mathbf{x}), l^z(\Omega\mu^2 t)\}; g^*, 1, 1), \tag{62}$$

where

$$\delta(n, \tilde{n}) = n\frac{d - 2 + \eta}{2} + \tilde{n}\frac{d + 2 + \tilde{\eta}}{2}. \tag{63}$$

In the following, to simplify the notation, we will set the constants $\mu$ and $\Omega$ to 1.

Let us consider in more detail the scaling forms of the two-point critical correlation function $C_{\mathbf{q}}(t, s)$ and response function $R_{\mathbf{q}}(t, s)$, with $t > s$. From equations (62) and (63), using TTI, one has

$$\begin{aligned} C_{\mathbf{q}}(t, s) &\equiv \int \mathrm{d}^d x\, \mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{x}} G_{2,0} = l^{\eta-2} f_C(l^{-1}\mathbf{q}, l^z(t - s)), \\ R_{\mathbf{q}}(t, s) &\equiv \Omega \int \mathrm{d}^d x\, \mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{x}} G_{1,1} = l^{\eta+z-2} f_R(l^{-1}\mathbf{q}, l^z(t - s)). \end{aligned} \tag{64}$$

If we choose $l = (t-s)^{-1/z}$, with $t - s \gg 1$, these scaling forms may be rewritten as ($q = |\mathbf{q}|$)

$$\begin{aligned} C_{\mathbf{q}}(t, s) &= (t - s)^{(2-\eta)/z} F_C^{\mathrm{eq}}(q(t - s)^{1/z}), \\ R_{\mathbf{q}}(t, s) &= (t - s)^{(2-\eta-z)/z} F_R^{\mathrm{eq}}(q(t - s)^{1/z}), \end{aligned} \tag{65}$$

where $F_R^{\mathrm{eq}}(x)$ and $F_C^{\mathrm{eq}}(x)$ are regular functions for small argument. The FDT provides a differential relation between $F_R^{\mathrm{eq}}(x)$ and $F_C^{\mathrm{eq}}(x)$.

By means of RG, one reduces the problem of calculating universal quantities to the evaluation of RG functions at the IR-stable fixed point. In order to find such a fixed point, a useful tool is the $\epsilon$ expansion. It is based on the observation that $d = 4$ is a special dimension for the model we are considering, in fact using the definition of the $\beta$ function we get

$$\beta(g) = \mu\partial_\mu g|_0 = -(4 - d)\left(\frac{\mathrm{d}\log g Z_g}{\mathrm{d}g}\right)^{-1}. \tag{66}$$

Being $Z_g = 1 + O(g)$, the fixed point $g^* = 0$ is IR stable for $d > 4$ and unstable in the opposite case. Thus, for $d > 4$ the critical behaviour is governed by the fixed-point $g^* = 0$ (Gaussian fixed point), corresponding to the Gaussian model. For $d < 4$ this is no longer true. However, adiabatically decreasing the dimension from four dimensions, the new IR stable fixed point is expected to be close to the Gaussian one, i.e., $g^* = O(\epsilon)$, where $\epsilon = 4 - d$. As originally pointed out in a seminal paper by Wilson and Fisher [79], one can perform a double expansion of RG functions in terms of $g$ and $\epsilon$ and find the zero(es) of the $\beta$ function as series in $\epsilon$. Finally, the critical exponents and other universal quantities are obtained expanding in $\epsilon$ the corresponding RG functions evaluated at the stable fixed point.

This procedure allows one to obtain the critical quantities as series in $\epsilon$. The analytical continuation of such series at physical dimensions $d = 3, 2$ (i.e., $\epsilon = 1, 2$) may not seem so straightforward. However, the validity of this continuation to $\epsilon = 1, 2$ is corroborated by the very good agreement of the $\epsilon$-expansion estimates (obtained applying resummation techniques based on Borel resummation [30]) with other theoretical and experimental values [30, 67].

*3.5.2. Non-equilibrium renormalization and scaling.* Let us consider the case with a non-equilibrium initial condition. As explained in section 3.4, it can be studied by averaging the correlation functions with the probability distribution function of the initial condition, specified by $\exp\{-\mathcal{H}_0\}$ [24]. In turn this implies that expectation values in equation (29) have to be computed with a total dynamical functional given by $S[\varphi, \tilde{\varphi}] + \mathcal{H}_0$ (see equation (35)). As long as $\mathcal{H}_0$ has the same form as the static Hamiltonian $\mathcal{H}$, but with different bare couplings, the usual renormalizations are enough to render the theory finite [24, 25]. This is no longer true if, for instance, the initial state is an uncritical one like a high-temperature state with short-range correlations and a small initially prepared magnetization $\langle\varphi_0(\mathbf{x})\rangle = a(\mathbf{x})$, i.e.

$$\langle[\varphi_0(\mathbf{x}) - a(\mathbf{x})][\varphi_0(\mathbf{x}') - a(\mathbf{x}')]\rangle = \tau_0^{-1}\delta(\mathbf{x} - \mathbf{x}'). \tag{67}$$

The corresponding $\mathcal{H}_0[\varphi_0]$ is Gaussian

$$\mathcal{H}_0[\varphi_0] = \int \mathrm{d}^d x\, \frac{\tau_0}{2}[\varphi_0(\mathbf{x}) - a(\mathbf{x})]^2. \tag{68}$$

The canonical mass dimension of $\tau_0$ is $[\tau_0]_{\mathrm{can}} = 2$, thus the possible fixed points of $\tau_0$ can be $\pm\infty$ and 0. Since $\tau_0 = 0$ and $-\infty$ yield non-normalizable distributions, the fixed point of physical interest is $\tau_0 = \infty$ [24]. This corresponds in the language of surface critical behaviour to the ordinary transition [80]. Corrections due to finite value of $\tau_0$ will be irrelevant in the RG sense, thus one can set $\tau_0^{-1} = 0$ from the very beginning of the calculation, as long as the leading contribution does not vanish with $\tau_0^{-1} = 0$.

Following standard methods, the response and correlation functions may be obtained by a perturbative expansion of the functional weight $\exp\{-S[\varphi, \tilde{\varphi}] - \mathcal{H}_0[\varphi_0]\}$ in terms of the coupling constant $g_0$. From the technical point of view, the breaking of TTI does not allow the factorization of connected correlation functions in terms of one-particle irreducible ones, as usually done when TTI is not broken. As a consequence all the perturbative computations must be done in terms of connected functions only [80, 24].

The addition of $\mathcal{H}_0[\varphi_0]$ gives rise to new divergences in perturbation theory whenever $s$ approaches the 'time surface' located at $t_0 \equiv 0$. On the other hand, the bulk renormalization functions defined by equations (51) are not changed by the presence of $\mathcal{H}_0[\varphi_0]$, if MS is employed to renormalize the theory. It is possible to remove these new singularities by counterterms 'located' at that time surface, as it has been shown in [80] for the case of surface critical phenomena. Moreover, being the canonical mass dimension of time variables $-2$, the degree of divergence of a generic correlation function decreases by 2 for each vanishing

time argument. As a consequence the new renormalizations are required only in the case of two-point functions. A detailed analysis shows that once equilibrium theory has been renormalized according to equation (51), only one new renormalization constant is required to render finite both the correlation and the response functions. In the case of non-equilibrium model A dynamics, the new renormalization is

$$\tilde{\varphi}_0(\mathbf{x}) \mapsto (Z_0\tilde{Z})^{1/2}\tilde{\varphi}_0(\mathbf{x}), \tag{69}$$

and $Z_0$ is the new renormalization constant.

The scaling properties of connected correlation functions

$$G^{\tilde{n}_0}_{n,\tilde{n}} = \langle [\varphi]^n [\tilde{\varphi}]^{\tilde{n}} [\tilde{\varphi}_0]^{\tilde{n}_0} \rangle \tag{70}$$

may be exploited by using RG equations as in equilibrium. To renormalize $G^{\tilde{n}_0}_{n,\tilde{n}}$ one uses all the $Z$-factors previously introduced (equations (51) and (69)), obtaining

$$G^{\tilde{n}_0}_{n,\tilde{n}} \mapsto \overset{\circ}{G}{}^{\tilde{n}_0}_{n,\tilde{n}} = Z^{n/2}\tilde{Z}^{\tilde{n}/2}(\tilde{Z}Z_0)^{\tilde{n}_0/2} G^{\tilde{n}_0}_{n,\tilde{n}}, \tag{71}$$

where again the rhs is expressed in terms of renormalized quantities. The RG equations are derived from $\mu\partial_\mu \overset{\circ}{G}{}^{\tilde{n}_0}_{n,\tilde{n}}\big|_0 = 0$, in analogy with equation (53). To solve them, a new Wilson's function $\gamma_0 = \mu\partial_\mu \ln Z_0|_0$ has to be introduced. The leading scaling behaviour of the correlation functions (70) is determined as in equilibrium

$$G^{\tilde{n}_0}_{n,\tilde{n}}(\{\mathbf{x}, t\}) = l^{\delta(n,\tilde{n},\tilde{n}_0)} G^{\tilde{n}_0}_{n,\tilde{n}}(\{l\mathbf{x}, l^z t\}), \tag{72}$$

where now

$$\delta(n, \tilde{n}, \tilde{n}_0) = n\frac{d-2+\eta}{2} + \tilde{n}\frac{d+2+\tilde{\eta}}{2} + \tilde{n}_0\frac{d+2+\tilde{\eta}+\eta_0}{2}, \tag{73}$$

with the novel exponent $\eta_0 = \gamma_0(g^*)$, in terms of which the so-called initial-slip exponent $\theta = -\eta_0/(2z)$ is defined [24]. The scaling dimensions of the surface field are $[\tilde{\varphi}_0]_{\text{scal}} = (d+2+\tilde{\eta}+\eta_0)/2$.

According to the same procedure, the analysis has been done for various models. Model C dynamics (see sections 3.4.3 and 7) has been studied in [81], whereas Models E and G (see section (3.4.4)) have been studied in [82]. Model A dynamics of a tricritical point is investigated in [83]. In [84, 85], the weakly dilute Ising model has been considered in the case of uncorrelated impurities. The case of extended random defects has also been addressed [86]. Other interesting universality classes have been studied [87].

Let us consider in more details the scaling forms of the two-point critical correlation function $C_{\mathbf{q}}(t, s)$ and response function $R_{\mathbf{q}}(t, s)$, with $t > s > 0$ (i.e., with $t$ and $s$ in the 'bulk'). From equations (72) and (73) one has (we set $\Omega = 1$)

$$\begin{aligned} C_{\mathbf{q}}(t, s) &\equiv G^0_{2,0} = l^{\eta-2} C_{l^{-1}\mathbf{q}}(l^z t, l^z s), \\ R_{\mathbf{q}}(t, s) &\equiv G^0_{1,1} = l^{\eta+z-2} R_{l^{-1}\mathbf{q}}(l^z t, l^z s). \end{aligned} \tag{74}$$

Considering $l = (t-s)^{-1/z}$, these scaling forms may be rewritten as

$$\begin{aligned} C_{\mathbf{q}}(t, s) &= (t-s)^{(2-\eta)/z} \tilde{\mathcal{F}}_C(q(t-s)^{1/z}, s/t), \\ R_{\mathbf{q}}(t, s) &= (t-s)^{(2-\eta-z)/z} \tilde{\mathcal{F}}_R(q(t-s)^{1/z}, s/t), \end{aligned} \tag{75}$$

where, as a fundamental difference to equilibrium ones, they depend on the ratio $s/t$.

We observe that the scaling functions $\tilde{\mathcal{F}}_C$ and $\tilde{\mathcal{F}}_R$ just introduced are not expected to be regular when $s$ approaches the time surface, i.e., for $s \to 0$. To determine the functional form of the correlation functions when their arguments approach exceptional points, the short-distance expansion (SDE) must be used (see [30, 57] for general reference and [80] for applications to

surface critical phenomena). The starting point is a formal expansion of the fields $\varphi(\mathbf{x}, s)$ and $\tilde{\varphi}(\mathbf{x}, s)$ around $s = 0$. First of all one notes that, when inserted in correlation functions with bulk fields, the following relations hold [24, 25]

$$\varphi(\mathbf{x}, 0) = \varphi_0(\mathbf{x}) = 0 \qquad \text{and} \qquad \partial_s \varphi(\mathbf{x}, s)|_{s=0} \equiv \dot{\varphi}_0(\mathbf{x}) = 2\Omega\tilde{\varphi}_0(\mathbf{x}). \quad (76)$$

As a consequence, for small $s$, one can formally expand the fields as

$$\varphi(\mathbf{x}, s) \sim \phi(s)\tilde{\varphi}_0(\mathbf{x}) + \text{h.o.c.f.}, \quad (77)$$

$$\tilde{\varphi}(\mathbf{x}, s) \sim \tilde{\phi}(s)\tilde{\varphi}_0(\mathbf{x}) + \text{h.o.c.f.}, \quad (78)$$

where h.o.c.f stands for higher-order composite fields which are neglected if one is interested only in the leading contributions in the limit $s/t \to 0$. Inserting the relations (77) and (78) into the correlation functions and taking into account the scaling behaviour equation (72), one deduces that, at criticality [24, 25],

$$\phi(s) = a_C s^{1-\theta}, \qquad \tilde{\phi}(s) = a_R s^{-\theta}, \quad (79)$$

(i.e., $[\phi]_{\text{scal}} = [\varphi]_{\text{scal}} - [\tilde{\varphi}_0]_{\text{scal}} = -z(1-\theta)$ and $[\tilde{\phi}]_{\text{scal}} = [\tilde{\varphi}]_{\text{scal}} - [\tilde{\varphi}_0]_{\text{scal}} = z\theta$, with $[\text{time}]_{\text{scal}} = -z$) where $a_C$ and $a_R$ are two non-vanishing constants. Accordingly for small $s$ one has

$$\begin{aligned} C_{\mathbf{q}}(t, s) &= \phi(s)\langle\varphi(\mathbf{q}, t)\tilde{\varphi}_0(-\mathbf{q})\rangle, \\ R_{\mathbf{q}}(t, s) &= \tilde{\phi}(s)\langle\varphi(\mathbf{q}, t)\tilde{\varphi}_0(-\mathbf{q})\rangle. \end{aligned} \quad (80)$$

From equation (72), the scaling form of $\langle\varphi(\mathbf{q}, t)\tilde{\varphi}_0(-\mathbf{q})\rangle \equiv G^1_{1,0}(\{\mathbf{q}, t\}; r)$ can be determined and at criticality

$$G^1_{1,0}(\{\mathbf{q}, t\}; 0) = l^{-\theta z-(2-z-\eta)} G^1_{1,0}(\{l^{-1}\mathbf{q}, l^z t\}; 0). \quad (81)$$

Taking into account the previous three relations, the following conclusion may be drawn ($q = |\mathbf{q}|$)

$$\begin{aligned} C_{\mathbf{q}}(t, s) &= a_C t^{(2-\eta)/z}(t/s)^{\theta-1} f_C(qt^{1/z}), \\ R_{\mathbf{q}}(t, s) &= a_R t^{(2-z-\eta)/z}(t/s)^{\theta} f_R(qt^{1/z}). \end{aligned} \quad (82)$$

Comparing these forms with equation (75) one concludes that for $y \to 0$

$$\begin{aligned} \tilde{\mathcal{F}}_C(x, y) &\sim a_C y^{-\theta+1} f_C(x), \\ \tilde{\mathcal{F}}_R(x, y) &\sim a_R y^{-\theta} f_R(x). \end{aligned} \quad (83)$$

It is possible to rewrite equation (75) in terms of scaling functions $\tilde{F}_C(x, y)$ and $\tilde{F}_R(x, y)$ with a regular behaviour for $y \to 0$, i.e.,

$$C_{\mathbf{q}}(t, s) = (t-s)^{(2-\eta)/z}(t/s)^{\theta-1}\tilde{F}_C(q(t-s)^{1/z}, s/t), \quad (84)$$

$$R_{\mathbf{q}}(t, s) = (t-s)^{(2-\eta-z)/z}(t/s)^{\theta}\tilde{F}_R(q(t-s)^{1/z}, s/t). \quad (85)$$

These results refer to systems in the infinite-volume limit. Finite-size effects on the non-equilibrium dynamics have been studied in [88].

For $\mathbf{q} = 0$, the scaling forms (84) and (85) may be written as

$$R_{\mathbf{q}=0}(t, s) = A_R(t-s)^a(t/s)^{\theta} F_R(s/t), \quad (86)$$

$$C_{\mathbf{q}=0}(t, s) = A_C s(t-s)^a(t/s)^{\theta} F_C(s/t), \quad (87)$$

where $a = (2-\eta-z)/z$. We singled out explicitly the non-universal amplitudes $A_{R,C}$ by fixing $F_{R,C}(0) = 1$. With this normalization, $F_{R,C}$ are universal scaling functions. From the previous scaling forms one deduces that

$$\partial_s C_{\mathbf{q}=0}(t, s) = A_{\partial C}(t-s)^a(t/s)^{\theta} F_{\partial C}(s/t), \quad (88)$$

where the non-universal amplitude $A_{\partial C}$ has been defined in such a way that $F_{\partial C}(0) = 1$. The relation between $F_{\partial C}(x)$ and $F_C(x)$ can be easily worked out. The amplitudes are related by $A_{\partial C} = A_C(1-\theta)$.

Using equations (86) and (88) one finds that

$$\mathcal{X}_{\mathbf{q}=0}(t,s) \equiv \frac{R_{\mathbf{q}=0}(t,s)}{\partial_s C_{\mathbf{q}=0}(t,s)} = \frac{A_R F_R(s/t)}{A_{\partial C} F_{\partial C}(s/t)} \tag{89}$$

is a universal scaling function, being the ratio of two quantities ($R_{\mathbf{q}=0}(t,s)$ and $\partial_s C_{\mathbf{q}=0}(t,s)$) that have the same scaling dimensions. Furthermore, it is a function only of $s/t$ and not of $s$ and $t$ separately.

From equation (89) and using the results of section 2.4, one easily finds that

$$X^{\infty} = \lim_{s\to\infty}\lim_{t\to\infty} \frac{R_{\mathbf{q}=0}(t,s)}{\partial_s C_{\mathbf{q}=0}(t,s)} = \lim_{s/t\to 0} \mathcal{X}_{\mathbf{q}=0}(s/t) = \frac{A_R}{A_{\partial C}} = \frac{A_R}{A_C(1-\theta)}, \tag{90}$$

is a universal, dimensionless amplitude ratio in the sense of [69].

Fourier transforming equations (84) and (85) we obtain the scaling forms of the autoresponse and autocorrelation functions

$$R_{\mathbf{x}=0}(t,s) = \mathcal{A}_R(t-s)^{a-d/z}(t/s)^{\theta}\mathcal{F}_R(s/t), \tag{91}$$

$$C_{\mathbf{x}=0}(t,s) = \mathcal{A}_C s(t-s)^{a-d/z}(t/s)^{\theta}\mathcal{F}_C(s/t), \tag{92}$$

with $\mathcal{A}_R$, $\mathcal{A}_C$, $\mathcal{F}_R(y)$ and $\mathcal{F}_C(y)$ that are (in analogy with their counterpart in the real space) non-universal and universal, respectively.

Note that even $X_{\mathbf{x}=0}(t,s)$ is a function of $s/t$ only. This is an important difference compared to mean-field glassy models, where $X_{\mathbf{x}=0}$ can be written as a function of $C_{\mathbf{x}=0}(t,s)$ (see, e.g., [7]).

### *3.6. The effect of the initial condition and the irrelevance of* $\tau_0^{-1}$

In the previous section, by means of dimensional analysis, we concluded that $\tau_0^{-1}$ is an irrelevant variable and so it can be neglected as long as one is interested in the leading long-time behaviour. However, this is not possible when considering quantities that vanish for $\tau_0^{-1} = 0$. This is actually the case of the correlation function when the smaller time $s$ goes to zero, as it is clear from equations (87) and (92). These forms have been obtained assuming large $s$, but it is simple to show that due to the Dirichlet boundary condition in time (implicit in the assumption $\tau_0^{-1} = 0$) the correlation functions with the insertion of one $\varphi(\mathbf{x}, 0)$ vanish identically, see equation (76). In these cases the contribution depending on $\tau_0^{-1}$ is the leading one and so the scaling forms cannot be calculated setting $\tau_0^{-1} = 0$. It was shown that the insertion of a boundary field is equivalent to the insertion of $\tau_0^{-1}\tilde{\varphi}_0$ [25]:

$$\varphi(\mathbf{x}, 0) = \tau_0^{-1}\tilde{\varphi}_0(\mathbf{x}), \tag{93}$$

when inserted in a correlation function with bulk fields. Therefore, using equations (72) and (73), the scaling of the autocorrelation function for $s = 0$ and large $t$ turns out to be (neglecting higher-order corrections due to initial correlations, i.e., to finite $\tau_0$)

$$A(t) = C_{\mathbf{x}=0}(t,0) = \tau_0^{-1}\langle\varphi(\mathbf{x},t)\tilde{\varphi}_0(\mathbf{x})\rangle \simeq \tau_0^{-1}t^{-\delta(1,0,1)/z} = \tau_0^{-1}t^{\theta'-d/z}, \tag{94}$$

where we introduced the so-called magnetization initial-slip exponent, given by $\theta' = [d - \delta(1,0,1)]/z$ (see equation (73)). For model A dynamics it can be written as

$$\theta' = \theta + a = \theta + (2 - \eta - z)/z. \tag{95}$$

Equations (94) and (92) imply that for $t \gg s$, $C_{\mathbf{x}=0}(t,0) \sim \tau_0^{-1} t^{-d/z+\theta'}$ whereas $C_{\mathbf{x}=0}(t,s) \sim s^{1-\theta} t^{-d/z+\theta'}$, at the leading order in $\tau_0^{-1}$. In both the cases the contributions decrease as $t^{-d/z+\theta'}$, allowing for the identification of the autocorrelation exponent, defined as [46]

$$C_{\mathbf{x}=0}(t,s) \simeq t^{-\lambda/z} \qquad \text{for} \quad t \gg s, \tag{96}$$

so that

$$\lambda = d - \theta' z = d - \theta z - 2 + \eta + z. \tag{97}$$

Let us point out that the large-$t$ behaviour of $C_{\mathbf{x}=0}(t,0)$ and $C_{\mathbf{x}=0}(t,s)$ can, *a priori*, be different, giving rise to a temporal cross-over for finite $s$, as in the case of model B dynamics considered in section 6.

For the sake of completeness we comment that also autoresponse exponent is usually defined in the literature [89], through

$$R_{\mathbf{x}=0}(t,s) \sim t^{-\lambda_R/z} \qquad \text{for} \quad t \gg s. \tag{98}$$

For model A dynamics $\lambda_R$ equals $\lambda$. This is not the case in the more general case, see section 6.

### *3.7. Comments*

Some remarks are now in order concerning the particular choices of the noise and of the initial conditions.

If in equation (33) we fix $\zeta(\mathbf{x},t) = 0$, we obtain the deterministic evolution of an order parameter under a diffusion-like equation. In turn, the associated dynamical action has no quadratic term in $\tilde{\varphi}$. Physically this can be used to describe the evolution of a system quenched from a fully disordered initial state to a critical point at which the thermal noise is negligible. This should be the case of systems with $T_c = 0$. However, it is not clear under which conditions the microscopic models defined via a master equation (and hence stochastic) are in the same universality classes as the corresponding deterministic mesoscopic equations. Examples of different resulting universality classes have been provided [90]. The RG study and $\epsilon$ expansion of this deterministic model have been worked out in [66]. For our discussion, a relevant finding is that response and correlation functions scale in time with $\lambda \neq \lambda_R$, resulting in a trivial FDR. For this reason we will not further consider the deterministic evolution.

We have also assumed the noise to be Gaussian and white. A non-constant spectral density of the noise may also occur (in this case the noise is called 'coloured'). To our knowledge, the non-equilibrium behaviour under the effects of a coloured noise has been considered only for the random walk [91], corresponding to the Gaussian approximation of the $\mathbf{q} = 0$ mode in equation (33).

So far, we considered only the case of short-range correlations in the initial state. Long-range ones (for instance decaying as $\varsigma_0 |\mathbf{x}|^{-\rho}$ for large $\mathbf{x}$) may change the dynamic non-equilibrium universality class and in particular the initial-slip exponent $\theta$ and $X^\infty$. Decreasing the exponent $\rho$ of the long-range correlations, the canonical dimension of $\varsigma_0$, that is negative for large enough $\rho$, may become positive. In this case $\varsigma_0$ turns to a relevant parameter in the RG sense and one expects that the stable fixed point and consequently the critical properties get modified. Exact calculations for the spherical model [89] and for the one-dimensional Ising model [92] substantiate this picture.

Finally, a scaling correlation function of the form of equation (92) was also found in models of self-organized criticality [93].

### 3.8. Scaling of high-order observables

In this section, we briefly discuss the scaling behaviour of observables that are powers of the order parameter at the same spacetime point (composite operators in FT language). More details can be found in [23].

Let us consider an observable $\mathcal{O}$ with $h_{\mathcal{O}}$ as a conjugate field (e.g., $\mathcal{O}$ is the energy density and $h_{\mathcal{O}}$ the temperature) that couples to $\mathcal{H}$ according to $\mathcal{H} \mapsto \mathcal{H} - \beta h_{\mathcal{O}} \mathcal{O}$. As a consequence, the dynamical functional $S$ changes according to $\mapsto S - \beta h_{\mathcal{O}} \widetilde{\mathcal{O}}$, where the associated operator $\widetilde{\mathcal{O}}$ is given by

$$\widetilde{\mathcal{O}} = \int \mathrm{d}t\, \mathrm{d}^d x \tilde{\varphi}(\mathbf{x}, t) \mathcal{D} \frac{\delta \mathcal{O}}{\delta \varphi(\mathbf{x}, t)}. \tag{99}$$

The linear response of an observable $\mathcal{A}$ to a variation in the field $h_{\mathcal{O}}$ conjugated to $\mathcal{O}$ can be expressed as

$$\left. \frac{\delta \langle \mathcal{A} \rangle_{h_{\mathcal{O}}}}{\delta h_{\mathcal{O}}} \right|_{h_{\mathcal{O}}=0} = \beta \langle \mathcal{A} \widetilde{\mathcal{O}} \rangle, \tag{100}$$

where $\langle \cdot \rangle_{h_{\mathcal{O}}}$ stands for the average over the dynamics associated with the dynamical functional in the presence of $h_{\mathcal{O}}$.

We are interested in the scaling properties of the correlation functions of the form $\langle \mathcal{O} \widetilde{\mathcal{O}} \rangle$ and $\langle \mathcal{O} \mathcal{O} \rangle$. Using RG equations as in the previous sections, one obtains [23]

$$\begin{aligned} \langle \mathcal{O}(\mathbf{q}, t) \mathcal{O}(-\mathbf{q}, s) \rangle &= (t-s)^{a_{\mathcal{O}}+1} \hat{F}_C(q^z(t-s), s/t), \\ \langle \mathcal{O}(\mathbf{q}, t) \widetilde{\mathcal{O}}(-\mathbf{q}, s) \rangle &= (t-s)^{a_{\mathcal{O}}} \hat{F}_R(q^z(t-s), s/t), \end{aligned} \tag{101}$$

where $a_{\mathcal{O}} = (2 - \eta_{\mathcal{O}} - z)/z$ and $\eta_{\mathcal{O}}$ is the scaling dimension of the observable $\mathcal{O}$.

In the presence of a time surface the functions $\hat{F}_C(q^z(t-s), s/t)$ and $\hat{F}_R(q^z(t-s), s/t)$ are not regular in the limit $s/t \to 0$, as in the case of $\mathcal{O} = \varphi$ of the previous section. To extract the leading singularity in the ageing regime, a proper short-distance expansion should be considered. This can be easily worked out for $\mathcal{O}^{(m)}(t) = \varphi^m(t)$ (hence $\widetilde{\mathcal{O}}^{(m)}(t) \sim \tilde{\varphi}(t)\varphi^{m-1}(t)$). In this case the analysis is straightforward and the final scaling forms are [23]

$$\begin{aligned} \langle \mathcal{O}^{(m)}(\mathbf{q}, t) \mathcal{O}^{(m)}(-\mathbf{q}, s) \rangle &= s(t-s)^{a_{\mathcal{O}}} (t/s)^{-(m-1)+m\theta} F_C^{\mathcal{O}}(q^z(t-s), s/t) \\ \langle \mathcal{O}^{(m)}(\mathbf{q}, t) \widetilde{\mathcal{O}}^{(m)}(-\mathbf{q}, s) \rangle &= (t-s)^{a_{\mathcal{O}}} (t/s)^{-(m-1)+m\theta} F_R^{\mathcal{O}}(q^z(t-s), s/t), \end{aligned} \tag{102}$$

where the functions $F_C^{\mathcal{O}}$ and $F_R^{\mathcal{O}}$ are regular for $s/t \to 0$. Furthermore they are also universal once we fix the normalization for small arguments.

Let us focus on the scaling properties of the correlation and response functions of the two zero-momentum observables with $m = 2$:

$$E(t) \equiv \sum_{i=1}^{N} \int \mathrm{d}^d x\, \varphi_i^2(\mathbf{x}, t) = \sum_{i=1}^{N} \int (\mathrm{d}q) \varphi_i(\mathbf{q}, t) \varphi_i(-\mathbf{q}, t), \tag{103}$$

$$T_{i \neq j}(t) \equiv \int \mathrm{d}^d x\, \varphi_i(\mathbf{x}, t) \varphi_j(\mathbf{x}, t) = \int (\mathrm{d}q) \varphi_i(\mathbf{q}, t) \varphi_j(-\mathbf{q}, t). \tag{104}$$

$E$ and $T_{i \neq j}$ are usually referred to as the 'energy' and the quadratic tensor operator, respectively [23]. The corresponding response operators are

$$\begin{aligned} \widetilde{E}(t) &= \sum_{i=1}^{N} \int (\mathrm{d}q) 2\tilde{\varphi}_i(\mathbf{q}, t) \varphi_i(-\mathbf{q}, t) \\ \widetilde{T}_{i \neq j}(t) &= \int (\mathrm{d}q) [\tilde{\varphi}_i(\mathbf{q}, t) \varphi_j(-\mathbf{q}, t) + \varphi_i(\mathbf{q}, t) \tilde{\varphi}_j(-\mathbf{q}, t)]. \end{aligned} \tag{105}$$

In the following we will generically refer to them as $\mathcal{O}(t)$. In this case, the scaling dimension of $\mathcal{O}(t)$ being well known, the exponent $a_{\mathcal{O}}$ can be written in terms of the usual critical exponents [23]. The corresponding scaling forms for the correlation and response functions (the latter defined as in equation (100) with $\mathcal{A} = \mathcal{O}$) are

$$C^{\mathcal{O}}(t,s) = \langle \mathcal{O}(t)\mathcal{O}(s)\rangle = A_C^{\mathcal{O}} s(t-s)^{a_{\mathcal{O}}}(t/s)^{-1+2\theta} F_C^{\mathcal{O}}(s/t), \tag{106}$$

$$R^{\mathcal{O}}(t,s) = \langle \mathcal{O}(t)\widetilde{\mathcal{O}}(s)\rangle = A_R^{\mathcal{O}}(t-s)^{a_{\mathcal{O}}}(t/s)^{-1+2\theta} F_R^{\mathcal{O}}(s/t), \tag{107}$$

where the non-universal amplitudes $A_{C,R}^{\mathcal{O}}$ are chosen in such a way that $F_{C,R}^{\mathcal{O}}(0) = 1$. (As previously done, we have absorbed the factor $\beta^{-1}$ in the definition of $R^{\mathcal{O}}(t,s)$.) In terms of these quantities we can write the (zero-momentum) FDR as

$$\mathcal{X}_{\mathcal{O}}(t,s) \equiv \frac{R^{\mathcal{O}}(t,s)}{\partial_s C^{\mathcal{O}}(t,s)} \tag{108}$$

that, as in its analogous expression (89), depends only on the ratio $s/t$, and it is a universal function. In particular its long-time limit

$$\mathcal{X}_{\mathcal{O}}^{\infty} = \lim_{s\to\infty}\lim_{t\to\infty} \mathcal{X}_{\mathcal{O}}(t,s) = \frac{1}{2}\frac{A_R^{\mathcal{O}}}{(1-\theta)A_C^{\mathcal{O}}}, \tag{109}$$

is universal (analogously to equation (90)). These considerations hold in general for all the FDRs that can be obtained from equations (102).

## 4. Purely dissipative dynamics of an $O(N)$ model (model A)

One of the simplest non-trivial models displaying ageing after a quench to the critical point is a lattice spin model in $d$ dimensions with $O(N)$ symmetry, evolving according to a purely dissipative dynamics (no conservation laws). In the simplest instance, its Hamiltonian is given on the lattice by

$$\mathcal{H} = -\sum_{\langle \mathbf{ij}\rangle} \mathbf{s_i}\cdot\mathbf{s_j}, \tag{110}$$

where $\mathbf{s_i}$ is a $N$-component spin located at the lattice site $\mathbf{i}$, with $\mathbf{s_i}^2 = 1$. Here and in the following, the symbol $\langle \mathbf{ij}\rangle$ means that the sum runs on all nearest-neighbour pairs of lattice sites. When $N = 1$ the Hamiltonian (110) describes the Ising model, whereas for $N = 2$ the XY or planar model and for $N = 3$ the (isotropic) Heisenberg model. A purely dissipative dynamics for the lattice model (110) proceeds by elementary moves that amount to random changes in the direction of the spin $\mathbf{s_i}$ (spin-flip sampling). The transition rates can be arbitrarily chosen provided that the detailed-balance condition is satisfied. For analytical studies the most suited is the Glauber dynamics [76], which allows some exact solutions. Given its relative simplicity, this lattice model is the most studied and best understood. In the following, we review analytical and numerical results that can be compared with the field-theoretical (FT) ones presented in the next sections. In fact the coarse-grained continuum dynamics described by the Langevin equations (33) is expected to be in the same universality class as lattice models with $O(N)$ symmetry, short-range interactions and spin-flip dynamics [26]. All the reviewed determinations of the FDRs are summarized in tables 1 (magnetization FDR) and 2 (other observable FDRs) together with the FT estimates.

*One-dimensional Ising model.* In this case there is no finite temperature phase transitions, since $d = 1$ is the lower critical dimensionality of the Ising model, therefore $T_c = 0$. Analytical solutions for the critical response and correlation functions have been provided [94–97, 58]. From them, the critical exponents read $z = 2$ and $\theta = 1/2$ (as first shown in [98]). $X^{\infty} = 1/2$

**Table 1.** Estimates of $X^\infty$ in purely dissipative models quenched from a completely disordered initial state. [tr] and [hc] stand for simulations on the triangular and honeycomb lattices, respectively. When not indicated, the simulation is on the square lattice. Models belonging to the same universality class are marked with $^\dagger$ and $^\ddagger$.

| Model | Reference | Method | $X^\infty$ |
|---|---|---|---|
| Random walk | [16] | Exact | $1/2$ |
| Gaussian model | [16] | Exact | $1/2$ |
| $d$-dim spherical model | [17] | Exact | $1-2/d$ |
| 1-dim Ising model | [95, 94] | Exact | $1/2$ |
| 2-dim Ising model | [17] | MC | 0.26(1) |
| | [58] | MC | 0.340(5) |
| | [114] | MC | 0.33(2) |
| | [64] | MC | 0.33(1) |
| | [133] | MC | 0.330(5) |
| | [133] | MC[tr] | 0.326(7) |
| | [133] | MC[hc] | 0.330(8) |
| | [20] | FT $O(\epsilon^2)$ | 0.30(5) |
| 2-dim 3-state clock model$^\dagger$ | [133] | MC | 0.406(6) |
| | [133] | MC[tr] | 0.403(5) |
| | [133] | MC[hc] | 0.401(6) |
| 2-dim Ashkin–Teller model$^\dagger$ | [133] | MC | 0.405(10) |
| 2-dim 4-state Potts model$^\ddagger$ | [133] | MC | 0.475(10) |
| | [133] | MC[tr] | 0.470(12) |
| | [133] | MC[hc] | 0.467(7) |
| 2-dim Baxter–Wu model$^\ddagger$ | [133] | MC[tr] | 0.550(15) |
| 2-dim multispin Ising model$^\ddagger$ | [133] | MC | 0.467(10) |
| 3-dim Ising model | [17] | MC | $\simeq 0.40$ |
| | [20] | FT $O(\epsilon^2)$ | 0.429(6) |
| 3-dim $O(2)$ model | [20] | FT $O(\epsilon^2)$ | 0.416(8) |
| | [128] | MC | 0.43(4) |
| 3-dim $O(3)$ model | [20] | FT $O(\epsilon^2)$ | 0.405(10) |
| 3-dim random Ising model | [21] | FT $O(\sqrt{\epsilon})$ | $\simeq 0.416$ |

**Table 2.** Estimates of $X^\infty_{\mathcal{O}}$ for observables different from the magnetization. The meaning of the symbols [u], [l] and [q] in the FT estimates is explained in section 4.4.

| Model | Reference | Observable | Method | $X^\infty$ |
|---|---|---|---|---|
| Gaussian model | [23] | All | Exact | $1/2$ |
| $d$-dim spherical model | [59] | E | Exact | $\neq 1-2/d$ |
| | [59] | B, P | Exact | $1-2/d$ |
| 1-dim Ising model | [58] | E | Exact | 0 |
| | [58] | Other | Exact | $1/2$ |
| 2-dim Ising model | [58] | E | MC | 0.33(2) |
| 2-dim Ising model | [23] | E | FT | 0.33[u], 0.20[l] |
| 3-dim Ising model | [23] | E | FT | 0.39[u], 0.37[l] |
| 3-dim $O(2)$ model | [23] | E | FT | 0.37[u], 0.31[l], 0.22[q] |
| | [23] | T | FT | $\simeq 0.41$[u] |
| 3-dim $O(3)$ model | [23] | E | FT | 0.35[u], 0.30[l], 0.21[q] |
| | [23] | T | FT | $\simeq 0.41$[u] |

has also been found [94, 95] (actually this result was indirectly obtained several years before in [99]). Correlation and response functions of more complex observables have been studied

[37, 58, 97] and in particular the FDR for the energy has been found to be $X_E^\infty = 0$ at $T = 0$. The fact that $X^\infty \neq X_E^\infty$ is interpreted in [58] as an interplay between criticality and coarsening, a peculiarity of those models with $T_c = 0$. For many other observables, instead, the equality $X_{\mathcal{O}}^\infty = X^\infty$ between the critical FDRs holds [37, 58, 97]. In this context it must be recalled (as stated in previous sections) that the correspondence between the Glauber–Ising model and the Landau–Ginzburg dynamics is not obvious and they probably belong to two different universality classes (in fact they display different non-equilibrium exponent $\theta$ [90]).

*d-dimensional spherical and $O(\infty)$ models.* The static critical behaviour of the $O(N)$ model in the limit $N \to \infty$ is equivalent to that of the spherical model, defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{\langle \mathbf{ij}\rangle}(s_{\mathbf{i}} - s_{\mathbf{j}})^2, \tag{111}$$

where $s_{\mathbf{i}}$ are real numbers subjected to the constraint $\sum_{\mathbf{i}} s_{\mathbf{i}}^2 = L^d$ ($L$ being the linear dimension of the $d$-dimensional lattice, assumed for simplicity to be hypercubic). Indeed, Stanley proved [100] that the free energies of the two models are exactly the same. From this equality it follows that critical exponents, universal scaling functions etc are equal. The same equivalence also holds for equilibrium critical dynamics defined in the spherical model by means of a Langevin equation properly modified in order to prescribe a dynamics that is compatible with the spherical constraint. As far as we are aware, a proof of this equivalence has not yet been carried over to the non-equilibrium critical dynamics we are interested in. Let us recall that the free energies of the models in the presence of a spatial boundary (whose corresponding field theory looks very similar to the non-equilibrium one that we are considering) are not equal [101].

The spherical model has attracted a lot of attention, since its essentially Gaussian Hamiltonian makes it exactly solvable, even though the resulting critical behaviour is not mean-field-like in $2 < d < 4$, because of the spherical constraint. Even in the $O(\infty)$ model several quantities are exactly calculable because the fourth-order interaction term in the Hamiltonian (22) can be self-consistently decoupled according to $g/N(\varphi^2)^2 \to gC_{\mathbf{x}=0}(t)\varphi^2$ and the resulting Langevin equation (33) is linear and thus solvable [24]. The dynamical critical exponents for both $O(\infty)$ [24] and spherical models [17] are $z = 2$ and $\theta = 1 - d/4$, for $2 < d < 4$. The spherical FDR of the magnetization is $X_M^\infty = 1 - 2/d$ [17]. This result has been found in agreement with the direct calculation for the $O(\infty)$ model [23]. Recently Sollich [59] has determined several FDRs of quadratic operators (in the spin $s_{\mathbf{i}}$) that could be compared with FT results. He considered the bond energy observable $B_{\mathbf{i}} = \frac{1}{2}(s_{\mathbf{i}} - s_{\mathbf{j}})^2$, the product observable $P_{\mathbf{i}} = s_{\mathbf{i}}s_{\mathbf{j}}$ (with $\mathbf{i}$, $\mathbf{j}$ nearest neighbours) both in the real and in the momentum space, and the total energy $E$. The *exact* results of his analysis are that $X_P^\infty = X_B^\infty = X_M^\infty \neq X_E^\infty$. The scaling forms for these observables have also been derived and they agree with FT expectations equations (106) and (107) with $a_B = -1 - d/2$, $a_P = 1 - d/2$ and $a_E = d/2 - 3$ in $d < 4$ (and $a_E = a_P = a_B + 2 = 1 - d/2$ in $d > 4$). The calculation of response and correlation functions of high-order observables is rather cumbersome and it has not yet been carried out completely (see for details [23]).

*Two-dimensional Ising model.* The physically relevant cases with finite $N$ and $d = 2, 3$ are not analytically solvable and have been intensively investigated by means of Monte Carlo (MC) simulations. For the two-dimensional Ising model, the dynamical critical exponents are known with very high precision from lattice techniques (we recall that the static model is solvable, see, e.g., [102]). The most accurate determination of the critical exponent $z$ is $z = 2.1667(5)$ [103] (this number is obtained by means of equilibrium MC, since the exponent $z$ is the same in- and out-of-equilibrium). This value is in agreement with other less precise

simulations [104–107], but it is somehow bigger than the most precise FT estimates, i.e., the three-loop $\epsilon$ expansion [108], four-loop fixed dimension [109, 110] and the interpolation of two-loop result between one and four dimensions [111]. The most accurate value of the non-equilibrium critical exponent $\theta'$ has been obtained in [105], i.e., $\theta' = 0.191(3)$ that, using the scaling relation (95) leads to $\theta = 0.383(3)$, compatible with other MC results [46, 106, 107] and again slightly bigger than two-loop FT estimate [24].

Concerning the FDR, earlier investigations indicate $X^\infty = 0.26(1)$ [17]. Recent simulations provide results of increasing accuracy thanks to the improved methods that have been introduced to measure the quantities of interest. From the analysis of zero-momentum observables the very accurate estimate $X^\infty = 0.340(5)$ has been obtained [58] (the method employed in [58] has been questioned, see for details [112, 113]). Recently a new algorithm has been proposed by Chatelain [114] and subsequently revised in [115]. In standard simulations some integrated form of the response function is measured after the application of a magnetic field. The new algorithm instead allows one to measure the integrated linear response function without applying any magnetic field to the system, avoiding in such a way a possible crossover towards the nonlinear regime. By applying this algorithm, the result $X^\infty = 0.33(2)$ has been found [114]. The study of a more general model that belongs (apart one exceptional point in its phase diagram) to the Glauber–Ising universality class, even if it does not satisfy detailed balance, gives $X^\infty = 0.33(1)$ [64]. A few attempts have been made to measure the critical FDR for the energy $X_E^\infty$. In [64] the standard algorithm produced a signal too noisy to have any reliable result. By considering, instead, coherent observables $X_E^\infty = 0.33(2)$ has been obtained [58]. This result led the authors of [58] to put forward the conjecture that $X^\infty$ is the same for all the observables, according to the idea of the existence of a unique effective temperature, as it is the case for mean-field glassy models [5]. We comment that along the same lines of [114, 115], an alternative algorithm measuring directly the response function has been proposed [116], but it has only been tested in one-dimensional systems.

*Three-dimensional Ising universality class*. In this case, the numerical simulations are obviously less accurate than in two dimensions. The values of the static critical exponents are reviewed in [67] and will be not reported here, neither will for the $O(N)$ models with $N > 1$. The best determinations of the equilibrium dynamic critical exponent are $z = 2.032(4)$ [105] and $z = 2.055(10)$ [117]. They are not in agreement within three standard deviations, but they are in any case almost compatible with the result of less accurate simulations [118, 107]. Again the numerical values are a bit higher than the best FT estimates giving $z \simeq 2.02$ [108–111]. For the initial-slip exponent the best value is $\theta' = 0.104(3)$ [105] in agreement with less precise estimates [46, 118] and with the two-loop $\epsilon$ expansion [24, 25]. Through the scaling relation (95) and using the safe value $z = 2.04(2)$, this value of $\theta'$ leads to $\theta = 0.14(1)$. These values confirm that the $\epsilon$ expansion converges better in $d = 3$ than in $d = 2$.

Regarding the FDR, only the preliminary analysis of [17] is currently available, giving $X^\infty \simeq 0.40$.

*Two-dimensional XY universality class*. The non-equilibrium dynamics of this universality class has been studied in [16, 119–123]. The model is quite peculiar since, despite the absence of a spontaneous magnetization, it is critical for all temperatures below the so-called Kosterlitz–Thouless temperature $T_{\rm KT}$ (i.e., in the RG language, it has a line of fixed points) with a continuously varying critical exponent $\eta(T)$ that ranges from 0 to 1/4 as $T$ changes from 0 to $T_{\rm KT}$ [124]. Regarding the exponent $z$, all the approaches give $z = 2$, but with logarithmic corrections to the scaling depending upon the presence of vortices in the initial state [121]. For a quench from a high-temperature state to one of the critical points, numerical simulations have been so far inconclusive in determining non-equilibrium exponents. Indeed, in [120] the value of the critical exponent $\theta'$ was found to vary continuously along the line

of critical points, whereas in [123] it turned out to be temperature independent. A precise determination of this exponents surely deserves further analytical and numerical study. Even the determination of $X^\infty$ is problematic: in fact a standard FD plot provides the approximate relation $X^\infty = 1/2(T/T_{\rm KT})$ for $T \leqslant T_{\rm KT}$ [123], whereas a direct evaluation gives $X^\infty \simeq 0$, independently of $T$ [123]. Given this argument has not yet been settled we do not report these results in table 1. Let us mention that the FDR has also been studied also for the quench between two (critical) points below $T_{\rm KT}$ [122, 123, 125].

It is worth noting that Schehr and Le Doussal [126, 60] obtained a temperature-dependent FDR (similar to that just mentioned) $X^\infty = (2(1+{\rm e}^{\gamma_E}\tau)+O(\tau^2))^{-1}$, where $\tau = (T_g - T)/T_g$ ($T_g$ being the temperature of the glass transition in the model) in the glassy phase of the two-dimensional Cardy–Ostlund model [127], that also has a line of fixed points.

*Three-dimensional $O(N)$ models with $N > 1$.* These models have attracted a modest interest, mainly because the most relevant experimental realizations of $O(N)$ models belong to different dynamic universality classes as discussed in section 3.4.

To our knowledge, the three-dimensional XY model with purely dissipative dynamics has been numerically studied only in [128], using the approach discussed in [114, 115]. $\lambda/z \simeq 1.34$ was obtained, without providing independent determinations of the two exponents. A direct determination of the FDR $X(t,s)$ in the long-time limit turns out to be unreliable, since the numerical data are still affected by large statistical fluctuations. On the other hand, a linear extrapolation to the small-$s/t$ region provides the estimate $X^\infty = 0.43(4)$.

Concerning the $O(3)$ universality class, the determination of $z$ has been attempted only in [129], suggesting values slightly smaller than 2, i.e., $z = 1.96(6)$ [129]. On the other hand, FT studies unambiguously give $z > 2$ [108, 130, 131]. In fact, we point out that the early three-loop FT computation of $z$ [132] is affected by a numerical error, later corrected in [108], that led to the erroneous conclusion $z < 2$ in three dimensions. The correct value is $z = 2 + R\eta$, with a value of $R$ that is independent of $N$ up to three loops and reads $R = 0.726[1 - 0.1885\epsilon + O(\epsilon^2)]$ [108]. A more accurate numerical determination of $z$ surely deserves further analysis.

*Check of the universality of $X^\infty$.* The issue of universality of $X^\infty$ has been addressed in [133] by using the algorithm of [114, 115] for a variety of models, some of which are not related to the $O(N)$ models considered here. The long time non-equilibrium dynamics of the Ising, 3-state clock [133] and 4-state Potts [102, 133] models in two dimensions has been studied on square [sq], triangular [tr] and honeycomb [hc] lattices. The values found for $X^\infty$ are compatible with each other and with those already reported in the literature. For the Ising model it has been found $X^\infty[{\rm sq}] = 0.330(5)$, $X^\infty[{\rm tr}] = 0.326(7)$ and $X^\infty[{\rm hc}] = 0.330(8)$ on square, triangular and honeycomb lattice, respectively. For the 3-state clock model, instead, $X^\infty[{\rm sq}] = 0.406(6)$, $X^\infty[{\rm tr}] = 0.403(5)$ and $X^\infty[{\rm hc}] = 0.401(6)$, whereas for the 4-state Potts model $X^\infty[{\rm sq}] = 0.475(10)$, $X^\infty[{\rm tr}] = 0.470(12)$ and $X^\infty[{\rm hc}] = 0.467(7)$. These findings support the universality of $X^\infty$. Moreover, Chatelain measured $X^\infty$ for other models on the square lattice, which at least in equilibrium are expected to belong to the same universality class as the previously mentioned models. Namely, he investigated the Ashkin–Teller model [102, 133] at the point of its phase diagram where it belongs to the same universality class as the 3-state clock model, finding $X^\infty = 0.405(10)$, in perfect agreement with the previous results. Then he considered the multispin Ising model [133] and the Baxter–Wu model [102, 133] (defined on the triangular lattice and belonging to the same universality class as the 4-state Potts model) finding $X^\infty = 0.467(10)$ and $X^\infty = 0.550(15)$, respectively (note that, to our knowledge, the latter value is the first evidence for $X^\infty > 1/2$ in ferromagnetic models quenched at their critical points from an initially uncorrelated state). Whereas the former value is in perfect agreement with the result for the 4-state Potts model,

the latter is definitely incompatible with it. This can be a signal of a non-universality of $X^\infty$ (according to [133], even the dynamical exponent $\theta$ seems to depend on the specific lattice model, within the same dynamic universality class), or of the existence of a different (non-equilibrium) universality class to which the Baxter–Wu model belongs, or simply of large cross-over effects that make particularly difficult to reach the asymptotic scaling behaviour. It is still unclear whether these possibilities can explain the (somehow) surprising results of [133]. Further investigations and cross-checks are surely required.

### *4.1. Field-theoretical approach*

The coarse-graining of the lattice models specified by the Hamiltonian (110) with a Glauber-like dynamics leads to the purely dissipative dynamics of a $N$-component field $\varphi(\mathbf{x}, t)$ (model A of [26]), described by the stochastic Langevin equation (33), where $\mathcal{H}[\varphi]$ is the $\varphi^4$ Landau–Ginzburg Hamiltonian given in equation (22). The equilibrium correlation functions, generated by the Langevin equation (33) and averaged over the noise $\zeta$, can be obtained, as summarized in section 3.3, by means of the field-theoretical action (35). The problem is completely specified once equation (68) is assumed to provide the statistical distribution of the initial condition (with $\tau_0^{-1} = 0$ given we are interested only in the leading scaling behaviour, and $a(\mathbf{x}) = 0$). The propagators (i.e., the Gaussian two-point correlation and response functions) are obtained in a standard way, by considering the quadratic part of the action:

$$R^G_{\mathbf{q}}(t, s) = \Omega\langle\tilde{\varphi}_i(\mathbf{q}, s)\varphi_j(-\mathbf{q}, t)\rangle_G = \Omega\delta_{ij}\theta(t - s)\,\mathrm{e}^{-\Omega(\mathbf{q}^2+r_0)(t-s)}, \tag{112}$$

$$C^G_{\mathbf{q}}(t, s) = \langle\varphi_i(\mathbf{q}, s)\varphi_j(-\mathbf{q}, t)\rangle_G = \frac{\delta_{ij}}{\mathbf{q}^2 + r_0}\left[\mathrm{e}^{-\Omega(\mathbf{q}^2+r_0)|t-s|} - \mathrm{e}^{-\Omega(\mathbf{q}^2+r_0)(t+s)}\right], \tag{113}$$

where the subscript '$G$' is used to remind that we are considering the Gaussian approximation. The response function (112) is the same as in equilibrium. Equation (113) reduces, instead, to the equilibrium form when $\mathbf{q} \neq 0$ and both times $t$ and $s$ go to infinity while $t - s$ is kept fixed. This ensures that quasi-equilibrium (in the sense discussed in section 2.5) holds for the order parameter, at least within the Gaussian approximation.

### *4.2. Gaussian fluctuation–dissipation ratios*

In the Gaussian model (equation (22) with $g_0 = 0$) the response and correlations functions are exactly known and therefore the FDR can be easily evaluated. From equations (112), (113) and definition (18) one finds

$$\mathcal{X}_{\mathbf{q}}(t, s) = \frac{R_{\mathbf{q}}}{\partial_s C_{\mathbf{q}}} = \frac{1}{1 + \mathrm{e}^{-2\Omega(\mathbf{q}^2+r_0)s}}, \tag{114}$$

where we set $s < t$ and removed the subscript '$G$' since in this section we are only concerned with the Gaussian approximation. If the theory is non-critical ($r_0 > 0$), the limit of the FDR for $s \to \infty$ is 1 for all values of $\mathbf{q}$, in agreement with the idea that in the high-temperature phase all the fluctuating modes have a finite equilibration time. Equilibrium is hence recovered and the FDT applies. In the critical theory $r_0 = 0$, if $\mathbf{q} \neq 0$, the limit ratio is again equal to 1, whereas for $\mathbf{q} = 0$ one has $\mathcal{X}_{\mathbf{q}=0}(t, s) = \frac{1}{2}$, independently of $s$ and $t$. This shows that the only mode that 'does not relax' to the equilibrium is the zero mode in the critical limit. This picture has been confirmed by a one-loop computation [19].

In order to understand why measuring observables in momentum space is more effective than doing the same in real space [58], it is instructive to look at the FDR in real space. Computing the Fourier transform of equations (112) and (113), one finds (with $\Omega = 1$)

$$R_{\mathbf{x}}(t,s) = \int (\mathrm{d}q)\, \mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{x}}\, \mathrm{e}^{-(\mathbf{q}^2+r_0)(t-s)} = \frac{1}{(4\pi)^{d/2}}(t-s)^{-d/2}\, \mathrm{e}^{-r_0(t-s)-\mathbf{x}^2/[4(t-s)]}, \tag{115}$$

$$C_{\mathbf{x}}(t,s) = \int (\mathrm{d}q)\, \mathrm{e}^{i\mathbf{q}\mathbf{x}} \frac{\mathrm{e}^{-(\mathbf{q}^2+r_0)(t-s)} - \mathrm{e}^{-(\mathbf{q}^2+r_0)(t+s)}}{\mathbf{q}^2 + r_0}. \tag{116}$$

The derivative of the $C_{\mathbf{x}}(t,s)$ with respect to $s$ is

$$\partial_s C_{\mathbf{x}}(t,s) = \frac{1}{(4\pi)^{d/2}}\Big[(t-s)^{-d/2}\, \mathrm{e}^{-r_0(t-s)-\frac{\mathbf{x}^2}{[4(t-s)]}} + (t+s)^{-d/2}\, \mathrm{e}^{-r_0(t+s)-\frac{\mathbf{x}^2}{[4(t+s)]}}\Big]. \tag{117}$$

Therefore the FDR is

$$X_{\mathbf{x}}^{-1}(t,s) = 1 + \left(\frac{t-s}{t+s}\right)^{d/2} \mathrm{e}^{-2r_0 s + \frac{\mathbf{x}^2 s}{[2(t^2-s^2)]}}. \tag{118}$$

Again, for fixed $\mathbf{x}$ and $r_0 \neq 0$, in the ageing regime ($t, s \to \infty$, fixed $s/t$) this quantity reaches the asymptotic value $X_{\mathbf{x}}^{\infty} = 1$, and the system equilibrates. Instead for $r_0 = 0$ and $t, s \to \infty$ with fixed $s/t$ it reduces to

$$X_{\mathbf{x}}^{-1}(t,s) = 1 + \left(\frac{t-s}{t+s}\right)^{d/2}, \tag{119}$$

explicitly showing the crossover from the equilibrium value 1 at $s/t = 1$ to the pure ageing behaviour $X^{\infty} = 1/2$ for $t \gg s$ (apart from $d = 0$ where the model is a random walk).

The above considerations illustrate that $X_{\mathbf{x}}^{\infty}$ displays quite a long transient, lasting for a time of the order of the age of the system, before reaching its asymptotic value. This transient is absent in $\mathcal{X}_{\mathbf{q}=0}$, resulting in a more effective determination of $X^{\infty}$. As we shall see, the corrections to $\mathcal{X}_{\mathbf{q}}(t,s)/X^{\infty}$ due to the interaction term $g_0$ are very small in magnitude (at least for model A dynamics of $O(N)$ models), making this remark still valid.

For the Gaussian theory, the FDR can easily be computed for a generic observable. In particular it was shown [23] that the FDR for a set of local one-point observables is always equal to $1/2$ in the long-time limit. This argument is very simple and we briefly recall it here: let us consider the operators of the form $\mathcal{O}_{i,n} = \partial^i \varphi^n$. The critical response and correlation functions of the order parameter in real space are given by equations (115) and (116). The two-point correlation function of $\mathcal{O}_{0,n}$ is given by the $(n-1)$-loop diagram with the two points connected by $n$ correlation lines (see [23] for details). In real space its expression is simply given by the product of the $n$ correlators:

$$C_{\mathbf{x}}^{\mathcal{O}}(t,s) = c_n [C_{\mathbf{x}}(t,s)]^n, \tag{120}$$

($c_n$ is the combinatorial factor associated with the diagram) whose derivative is

$$\partial_s C_{\mathbf{x}}^{\mathcal{O}}(t,s) = c_n n [C_{\mathbf{x}}(t,s)]^{n-1} \partial_s C_{\mathbf{x}}(t,s). \tag{121}$$

Analogously, the response function is obtained from that one contributing to the correlation function by replacing an order-parameter correlator with a response function:

$$R_{\mathbf{x}}^{\mathcal{O}}(t,s) = c_n n [C_{\mathbf{x}}(t,s)]^{n-1} R_{\mathbf{x}}(t,s), \tag{122}$$

where the factor $n$ comes from the response operator $\widetilde{\mathcal{O}}_{0,n} = n\mathcal{O}_{0,n-1}\tilde{\varphi}$ (see equation (99)). Note that the combinatorial factor $c_n$ is the same as for the correlation function. The associated FDR is

$$X_{\mathbf{x}}^{\mathcal{O}}(t,s) \equiv \frac{R_{\mathbf{x}}^{\mathcal{O}}(t,s)}{\partial_s C_{\mathbf{x}}^{\mathcal{O}}(t,s)} = \frac{R_{\mathbf{x}}(t,s)}{\partial_s C_{\mathbf{x}}(t,s)} \equiv X_{\mathbf{x}}(t,s). \tag{123}$$

Therefore the FDR of powers of the order parameter in real space is equal, for all times, to the FDR of the order parameter itself.

Before considering the effects of the derivatives, let us consider the previous relation in momentum space. Remembering that the product of two functions after a Fourier transformation becomes a convolution, one has

$$R^{\mathcal{O}}_{\mathbf{q}}(t,s) = c_n n (C * \cdots * C * R)_{\mathbf{q}}, \tag{124}$$

$$\partial_s C^{\mathcal{O}}_{\mathbf{q}}(t,s) = c_n n (C * \cdots * C * C')_{\mathbf{q}}, \tag{125}$$

where $*$ is the convolution, $C$ and $R$ are in momentum space (with the time dependence understood), $\cdots$ means $n-1$ times and $C'_{\mathbf{q}} = \partial_s C_{\mathbf{q}}$. For $\mathbf{q} = 0$ the previous relations become

$$R^{\mathcal{O}}_{\mathbf{q}=0}(t,s) = c_n n \int (\mathrm{d}p)(C * \cdots * C)_{\mathbf{p}} R_{-\mathbf{p}}, \tag{126}$$

$$\partial_s C^{\mathcal{O}}_{\mathbf{q}=0}(t,s) = c_n n \int (\mathrm{d}p)(C * \cdots * C)_{\mathbf{p}} C'_{-\mathbf{p}}. \tag{127}$$

Thus

$$\left[\mathcal{X}^{\mathcal{O}}_{\mathbf{q}=0}(t,s)\right]^{-1} = \frac{\int (\mathrm{d}p)(C * \cdots * C)_{\mathbf{p}} C'_{-\mathbf{p}}}{\int (\mathrm{d}p)(C * \cdots * C)_{\mathbf{p}} R_{-\mathbf{p}}} = \frac{\int (\mathrm{d}p)(C * \cdots * C)_{\mathbf{p}} R_{-\mathbf{p}} \mathcal{X}^{-1}_{-\mathbf{p}}}{\int (\mathrm{d}p)(C * \cdots * C)_{\mathbf{p}} R_{-\mathbf{p}}}, \tag{128}$$

i.e., the inverse of $\mathcal{X}^{\mathcal{O}}_{\mathbf{q}=0}$ is a weighted average of $\mathcal{X}^{-1}_{-\mathbf{p}}$, with weight $(C * \cdots * C)_{\mathbf{p}} R_{-\mathbf{p}}$. In the limit $t \to \infty, s \to \infty$ this weight is expected to be peaked around $\mathbf{p} = \mathbf{0}$, giving $\mathcal{X}^{\infty}_{\mathcal{O}} \equiv \lim_{s\to\infty} \lim_{t\to\infty} \mathcal{X}^{\mathcal{O}}_{\mathbf{q}=0}(t,s) = \lim_{s\to\infty} \lim_{t\to\infty} \mathcal{X}_{\mathbf{p}=0}(t,s) = X^{\infty}$. Thus, the relation between the different FDRs holds also when considering $\mathcal{O}_{0,n>1}$. Note that in momentum space, at variance with relation (123) holding between FDR in real space, only the long-time limit of the FDR reproduces the FDR for the fields.

Let us now account for the effect of the derivative. In momentum space these are simply multiplication by $\mathbf{q}^i$ for each insertion, modifying the weight in equation (119) by a factor $\mathbf{q}^{2i}$ which does not change our conclusion about the long-time limit. Since all correlation and response functions of all local operators can be written in terms of those of $\mathcal{O}_{i,n}$, this concludes the argument.

It is worth mentioning that for observables different from the order parameter, $\mathcal{X}^{\mathcal{O}}_{\mathbf{q}}(t,s)$ is expected to depend non-trivially on the times even in the Gaussian model. Hence the previous argument about the advantage of using observables in momentum space to determine $X^{\infty}$ does not apply straightforwardly.

### *4.3. Two-loop order parameter FDR*

The program of calculating the universal two-point functions of the order parameter, the associated non-universal amplitudes and the corresponding FDR has been worked out up to two loops in an $\epsilon$ expansion [19, 20]. The analytic manipulations required for the calculation are quite long and involved. We summarize here only the results, referring the reader interested in technical details to the original works [19, 20]. The obtained exponents $z$ and $\theta$ agree with the already known two-loop expressions [30, 24]. The scaling of the response function (see equation (86)) is characterized by the non-universal amplitude

$$A_R = 1 + \epsilon^2 \frac{3(N+2)}{8(N+8)^2} \left[ f(0) - 4\gamma_E \log \frac{4}{3} \right] + O(\epsilon^3), \tag{129}$$

and the universal function

$$F_R(v) = 1 + \epsilon^2 \frac{3(N+2)}{8(N+8)^2}[f(v) - f(0)] + O(\epsilon^3), \tag{130}$$

whereas the scaling function of the derivative with respect to $s$ of the correlation function is (cf equation (88))

$$F_{\partial C}(v) = 1 + \epsilon^2 \frac{3(N+2)}{8(N+8)^2}\left[2\log\frac{4}{3}\log\frac{1-v}{1+v} + \Psi(v)\right] + O(\epsilon^3). \tag{131}$$

The correlation function has been computed only up to one loop [19]. We do not report here the quite long expression of $A_{\partial C}$ that can be worked out from the results in [20]. In equations (129), (130) and (131) $f(v)$ and $\Psi(v)$ are regular functions for small $v$, defined for $0 \leqslant v < 1$. Explicit expressions can be found in [20]. For the present purposes it is enough to note that they are of order 1 up to quite large $v$, i.e., $v \sim 0.9$. In terms of $A_R$, $A_{\partial C}$ and $\theta$, the long-time limit of the critical FDR is obtained via equation (90):

$$\frac{(X^\infty)^{-1}}{2} = 1 + \frac{N+2}{4(N+8)}\epsilon + \epsilon^2 \frac{N+2}{(N+8)^2}\left[\frac{N+2}{8} + \frac{3(3N+14)}{4(N+8)} + c\right] + O(\epsilon^3), \tag{132}$$

with $c = -0.0415\ldots$ (its analytic expression is given in [20])

Let us comment on these results and compare them with those currently available. The calculation shows that fluctuations give corrections to the Gaussian value of $F_R$ and $F_{\partial C}$ of order $\epsilon^2$. Unfortunately these are extremely tiny, making the numerical and experimental detection very difficult. This is due to the factor $\epsilon^2[3(N+2)]/[8(N+8)^2]$ in front of $f(v)$ and $\Psi(v)$. For the Ising model, this factor is $\simeq 0.014$ for $d = 3$ and $\simeq 0.05$ for $d = 2$, i.e., the two-loop corrections to the universal scaling functions are about 1% and 5% of the leading term in $d = 3$ and $d = 2$ respectively. This is not surprising given that even for the equilibrium scaling functions, the corrections to the Gaussian results involve the same prefactor [134].

However, the tiny correction in $F_R(v)$ is very important since it disagrees with the prediction, based on the theory local scale invariance (LSI) [135], that $F_R(v) = 1$ *exactly*. The LSI is a proposed extension of conformal invariance to system with anisotropic scaling, e.g., to dynamics where time and space scale differently. Given the large success of conformal invariance (see, e.g., [57]) for isotropic scaling, it is very interesting to provide generalizations to the anisotropic case. For Glauber–Ising model in $d = 2, 3$, the agreement of LSI predictions for the integrated response function with numerical simulation is really remarkable [54]. However, measuring a more suitable integrated response function, a small deviation from LSI has been recently detected [136]. This deviation is in qualitative agreement with the FT prediction in equation (130). The failure of LSI is probably attributed to the limits of applicability of the theory [137, 125] and might be traced back to the fact that the term $\Omega\tilde{\varphi}^2$ (generated by a non-vanishing thermal noise) breaks the Galilei invariance of the action (35) [125]. However, it has been argued that LSI should give exact predictions whenever $z = 2$ [125], even in the presence of the thermal noise. Field-theoretical results agree with this observation (see also sections 7 and 8). In fact, in the $\epsilon$-expansion, $F_R(v)$ is Gaussian at least up to the order at which $z = 2$ (i.e., $O(\epsilon)$ in model A of $\varphi^4$ theory, $O(\epsilon^0)$ in model C, and $O(\sqrt{\epsilon^0})$ in model A of the dilute Ising model).

Let us remark that the time dependence of $\mathcal{X}_{\mathbf{q}=\mathbf{0}}(t, s)$ shows up only through the two-loop corrections appearing in equations (130) and (131). According to previous discussion, these are expected to be quantitatively small, making it difficult to measure the difference between $\mathcal{X}_{\mathbf{q}=\mathbf{0}}(t, s)$ and its asymptotic value $X^\infty$, as noted in [113].

The expression (132) of $X^\infty$ for general $N$ provides quantitative predictions for a large class of systems. To get some numerical estimates out of this two-loop expansion a direct

summation (Padé approximant [2, 0]) and an 'inverse' one (Padé approximant [0, 2]) has been performed in [20]. From them some general trends may be understood:

- decreasing the dimensionality, $X^\infty$ always decreases (at least up to $\epsilon = 2$);
- increasing $N$, $X^\infty$ decreases, approaching the exact result of the spherical model;
- for $N = \infty$ the curve of the [0, 2] approximant reproduces better than the [2, 0] approximant the exact result in any dimension.

The last point suggests the use of the [0, 2] value as estimate of $X^\infty$, also for physical $N$. As an *indicative error bar* the difference between the two approximants has been used [20]. Accordingly, the numerical predictions for various cases can be summarized as follows:

(a) *Ising model in two and three dimensions.* Using the procedure just outlined, the FT estimates are $X^\infty = 0.30(5)$ for the two-dimensional Ising model and $X^\infty = 0.429(6)$ for the three-dimensional one [20]. In two dimensions, the agreement with numerical results is quite satisfactory despite the fact that *a priori* such a low-order perturbative expansion is not expected to provide such nice results. The estimate $X^\infty = 0.30(5)$ is marginally compatible with the first numerical determination $X^\infty = 0.26(1)$ [17]. The agreement with the subsequent estimates reviewed in table 1, and giving $X^\infty \simeq 0.33$, is clearly improved. In three dimensions the preliminary investigation of [17] gives $X^\infty \simeq 0.40$ for the Ising model. Albeit quite inaccurate, it is in nice agreement with the FT estimate $X^\infty = 0.429(6)$. Further simulation of this model in three dimensions would provide an important test of the actual reliability of this quite precise FT estimate.

(b) *XY model in three dimensions.* The result for $X^\infty$ yields also the accurate prediction $X^\infty = 0.416(8)$ for the purely dissipative three-dimensional $XY$ universality class. Currently, its ageing properties have been investigated only in [128] where the value $X^\infty = 0.43(4)$ is reported. On the one hand this result is compatible, within the estimated error bars, with the FT estimate. On the other hand, its numerical accuracy is still not sufficient to distinguish the differences with the Ising and Heisenberg universality classes in three dimensions.

(c) *Heisenberg model in three dimensions.* For the model A Heisenberg universality class in three dimensions, the estimate of $X^\infty$ provided in [20] is $X^\infty = 0.405(10)$. Currently, no numerical results are available for this case.

### *4.4. Two-loop FDRs of quadratic operators*

The two-loop response and correlation functions of the energy and the tensor operator (see equations (103) and (104), respectively) were computed for generic times in [23]. Their general expression is very cumbersome and we refer the interested reader to the original reference. Here we report and discuss only the value of $\mathcal{X}_\mathcal{O}^\infty$ [23]:

$$\mathcal{X}_\mathcal{O}^\infty = \begin{cases} \dfrac{1}{2}\left(1 - \dfrac{2}{3}\dfrac{N+2}{N+8}\epsilon\right) + O(\epsilon^2), & \mathcal{O} = E, \\ \dfrac{1}{2}\left(1 - \dfrac{1}{12}\dfrac{3N+16}{N+8}\epsilon\right) + O(\epsilon^2), & \mathcal{O} = T. \end{cases} \tag{133}$$

Comparing these results with equation (132) one concludes that the long-time limit of the fluctuation–dissipation ratio depends on the particular observable chosen to compute it.

Let us discuss in more detail some specific cases:

- $N = \infty$. In this case one finds

$$X_E^\infty = \frac{1}{2}\left(1 - \frac{2}{3}\epsilon\right) + O(\epsilon^2), \tag{134}$$

which is clearly different from the expression for the FDR of the order parameter, as can be checked expanding $X_M^\infty = 1 - 2/d$ close to $d = 4$. We note that equation (134) agrees with the expansion close to $d = 4$ of the result for the total energy in the spherical model [59]. On the basis of combinatorial arguments one expects $X_T^\infty$ to coincide with $X_M^\infty$ for $N = \infty$ to all orders in $\epsilon$ [23]. This expectation is confirmed by the two-loop FT computation equation (133).

These findings agree with results obtained for the spherical model [59]. Unexpectedly, the observable $P$ of the spherical model (see previous section) cannot be naively identified with $\varphi^2$ for $N = \infty$. On the other hand $a_B = a_P - 2$ agrees with the naive identification of $B$ as the Laplacian of $P$ in the continuum limit. This fact calls for a more rigorous investigation of a possible correspondence between the two models, beyond the case of equilibrium dynamics.

- *Ising model*. The direct estimate (unconstrained) from the $O(\epsilon)$ series in equation (133) for $\epsilon = 2$ (giving $X_E^\infty[\mathrm{u}] \simeq 0.28$) is probably unreliable, as was the case for $X_M^\infty$, i.e., $X_M^\infty[1\mathrm{loop}] \simeq 0.42$ [19] whereas $X_M^\infty[2\mathrm{loops}] = 0.30(5)$ [20]. To obtain a more reliable result without computing the $O(\epsilon^2)$ term, it was proposed in [23] to constrain linearly the available one-loop series to assume the exactly known value for $d = 1$ (i.e., $\epsilon = 3$). Assuming a smooth behaviour in $\epsilon$ up to $\epsilon = 3$, this gives [23]

$$X_E^\infty[\mathrm{l}] = \frac{1}{2}\left(1 - \frac{\epsilon}{3}\right)\left[1 + \frac{\epsilon}{9} + O(\epsilon^2)\right], \tag{135}$$

which has the same $\epsilon$-expansion as equation (133), but it is expected to converge more rapidly to the correct result. From equation (135) for the two-dimensional Ising model one gets $X_E^\infty[\mathrm{l}] \simeq 0.20$, that is much lower than the value determined so far, $X_M^\infty \simeq 0.33$ [20, 114, 58, 64, 133]. However, a robust field-theoretical prediction of $X_E^\infty$ for the two-dimensional Ising model requires a difficult higher-loop computation. For the three-dimensional Ising model equation (135) gives $X_E^\infty[\mathrm{l}] \simeq 0.37$, to be compared with the direct estimate $X_E^\infty[\mathrm{u}] \simeq 0.39$. Note that, as usual in $d = 3$, the spreading of the different estimates is much smaller, signalling a higher reliability of these predictions.

- *$O(N)$ model with $N > 1$*. From equation (133) predictions can be obtained for the purely dissipative three-dimensional $O(N)$ models with arbitrary $N$, again considering constrained analyses at the lower-critical dimension, which is $d_{\mathrm{lcd}} = 2$ in this case. For $N > 2$, $X_E^\infty(d = 2) = 0$, since for $d = 2$ these systems are in the coarsening regime. In [23] this was assumed generically for $N \geqslant 2$. Within this assumption a linear constraint gives [23]

$$X_E^\infty[\mathrm{l}] = \frac{1}{2}\left(1 - \frac{\epsilon}{2}\right)\left[1 + \frac{16 - N}{6(N + 8)}\epsilon + O(\epsilon^2)\right]. \tag{136}$$

On the other hand, the exact results for the spherical model [59] suggest that the approach to $d = 2$ is quadratic rather than linear. For this reason, also the quadratic constraint [23]

$$X_E^\infty[\mathrm{q}] = \frac{1}{2}\left(1 - \frac{\epsilon}{2}\right)^2\left[1 + \frac{20 + N}{3(N + 8)}\epsilon + O(\epsilon^2)\right], \tag{137}$$

was implemented. Comparing with the result for the spherical model, it was shown that for $N = \infty$ equation (137) provides a better approximation of the exact result than equation (136). For the three-dimensional $XY$ model ($N = 2$) one gets $X_E^\infty[\mathrm{u}] \simeq 0.37$ from direct estimate, $X_E^\infty[\mathrm{l}] \simeq 0.31$ from linear constraint and $X_E^\infty[\mathrm{q}] \simeq 0.22$ from the quadratic one. For the three-dimensional Heisenberg model ($N = 3$) one finds, instead, $X_E^\infty[\mathrm{u}] \simeq 0.35$, $X_E^\infty[\mathrm{l}] \simeq 0.30$ and $X_E^\infty[\mathrm{q}] \simeq 0.21$.

Even the results with constraints are rather scattered, making it difficult to provide firm estimates in $d = 3$. However, the conclusion $X_E^\infty < X_M^\infty$ for all $2 < d < 4$ is quite robust. Furthermore the difference between the two FDRs should be large enough to be observed in three-dimensional Monte Carlo simulations. The analysis of the non-equilibrium behaviour within the $\tilde{\epsilon} = d - 2$ expansion [30, 131] may clarify which, between the linear and the quadratic constraint, is the proper one close to $d = 2$.

We note that $X_T^\infty$ is very close to $X_M^\infty$ making the numerical detection of such a difference probably very difficult.

Let us contrast the FT results of [23] with those obtained for the one-dimensional Ising model [58, 97] and for the spherical model [59], where $X_{\mathcal{O}}^\infty = X_M^\infty$ for all the observables $\mathcal{O}$, except for the total energy. On the other hand, one could doubt that the energy operator is not as suited as others to define the effective temperature, being conjugated to the temperature of the bath but not to the actual one (if any) of the system. Nevertheless FT results show that there is at least one further operator, namely $T_{ij}$, having $X_T^\infty \neq X_M^\infty, X_E^\infty$. This supports the idea that a unique effective temperature cannot be defined for this class of models [23].

We recall that in disagreement with FT calculation, the MC simulation of the two-dimensional Ising model apparently gives $X_E^\infty = X_M^\infty$ [58]. Probably, a more accurate measure of $X_E^\infty$ is required to detect the difference (if any in $d = 2$) between $X_E^\infty$ and $X_M^\infty$.

## 5. Surface behaviour

It is a well-known experimental fact and theoretical result that the presence of surfaces influences both the static and dynamic equilibrium behaviour of critical systems [80, 138–143]. Upon approaching the critical point, local quantities such as the order parameter may display algebraic singularities at the surface differing from those characteristic of the bulk behaviour. Accordingly, one usually introduces *surface* critical exponents [80] in addition to the standard bulk ones. In the generic case, the former cannot be expressed in terms of the latter.

### *5.1. The semi-infinite $O(N)$ model: statics and dynamics*

As an illustrative example, let us consider the influence of a planar surface on the model (110) defined on a semi-infinite hypercubic lattice. Its Hamiltonian is given by

$$\mathcal{H}_{\infty/2} = -\sum_{\langle \mathbf{ij}\rangle \in \text{bulk}} \mathbf{s_i} \cdot \mathbf{s_j} - r \sum_{\langle \mathbf{ij}\rangle \in \text{surface}} \mathbf{s_i} \cdot \mathbf{s_j}, \tag{138}$$

where $r > 0$ is the coupling between nearest-neighbouring spins located at the surface. Because of the missing bonds, these spins experience an effective local magnetic field which is smaller than in the bulk, at least for sufficiently small $r$. Accordingly, one expects that the surface is not able to order independently of the bulk. Upon approaching the bulk critical point at the critical temperature $T_c$, the surface and the bulk order simultaneously. This defines the so-called *ordinary transition* (OT) and the associated surface critical behaviour is characterized by a given set of surface critical exponents [80]. On the other hand, upon increasing $r$ at fixed $T > T_c$, it is possible to overcompensate the missing bonds at the surface. As a consequence, the surface may order in the presence of a disordered bulk, provided that $d - 1$ is larger than the lower critical dimension of the model. For $r$ greater than the threshold value $r_{\text{sp}}$ there is a finite surface transition temperature $T_s(r) > T_c$ at which the so-called *surface transition* takes place. The surface critical behaviour of the surface transition is the same as the bulk critical one of the $(d-1)$-dimensional system. For $T_s(r) > T > T_c$ the system is characterized by an ordered surface and a disordered bulk. Upon further decreasing $T$, the bulk orders at $T_c$ in the

presence of an already ordered surface. This marks the so-called *extraordinary transition*. In the $(T, r)$-plane the three lines of phase transitions meet at the so-called multicritical *special transition* (SpT).

The same scenario is valid in equilibrium and non-equilibrium dynamics in semi-infinite systems: the dynamical behaviour is affected by the presence of the surface and depends on the surface universality class. On the other hand, no new and independent exponents show up in addition to the static surface and static and dynamic bulk ones. This is true in the case of both equilibrium [141–143] and non-equilibrium evolution [144–146].

The influence of a surface on the dynamical behaviour following a quench from the high-temperature phase to the critical point has been investigated in [147]. In particular the surface autocorrelation function and the associated surface autocorrelation exponent $\lambda_{c,1}$ are determined analytically at the OT and SpT for the model A dynamics of the $O(N)$ model up to the first order in the $\epsilon$-expansion and exactly for the $O(N)$ model with $N \to \infty$. MC simulations are instead used to investigate the same quantities at the OT of the two-dimensional Ising model with Glauber dynamics. This problem was reconsidered in [145], showing that the surface autocorrelation exponent is not a new dynamical quantity as claimed in [147] but it can be expressed in terms of the bulk autocorrelation exponent $\lambda_c$ and surface and bulk static critical exponents. The two-time surface autocorrelation function $C_1(t, s)$ was studied in [146]. By using scaling arguments, it was shown that a novel short-time behaviour (referred to as 'cluster dissolution') is expected in some cases, among which the three-dimensional Ising model at OT. In this case a non-algebraic decay of correlations is expected: $\ln C_1(t, s) \sim -(t - s)^{\kappa}$ for $t - s \ll s$ where the exponent $\kappa$ is given in terms of already known static surface and dynamic bulk exponents. This scenario was confirmed by MC simulation [146].

Previous studies have been extended to the response function and to the full dynamic scaling regime in [148]. The effects of the surface on the ageing properties of a critical systems are investigated via MC simulations at the OT of the two-dimensional Ising model and at the OT and SpT of the three-dimensional one. In addition the Hilhorst–van Leeuwen model is also considered [148]. The scaling behaviour observed in all these models agrees with scaling arguments, both for the surface autocorrelation function $C_1(t, s)$ and for the integral of the surface autoresponse function $R_1(t, s)$. In terms of $R_1$ and $C_1$ the surface FDR can be defined as

$$X_1(t, s) = \frac{T R_1(t, s)}{\partial_s C_1(t, s)}. \tag{139}$$

$X_1$ is analogous to the FDRs previously introduced. As in the bulk, it is possible to show that $X_1(t, s)$ is a universal function and its long-time limit $X_1^{\infty}$ is a universal amplitude ratio. As such, it is expected to depend on the specific surface universality class. In [148] it was found that $X_1^{\infty} = 0.37(1)$ for the two-dimensional Ising model (OT) whereas for the three-dimensional Ising model $X_1^{\infty} = 0.59(2)$ at OT and $X_1^{\infty} = 0.44(2)$ at SpT.

From these results it is quite clear that $X_1^{\infty} \neq X^{\infty}$ (cf table 1) and in particular that $X_1$ depends on the surface universality class, as expected.

It is worth noting that MC simulations suggest that $X_1^{\infty}$ can be greater than $\frac{1}{2}$ at the OT point, as a difference with the bulk, where $X^{\infty}$ is always $\leqslant \frac{1}{2}$ for $O(N)$ models.

### *5.2. The semi-infinite $O(N)$ universality class: field-theoretical approach*

In the previous sections, we have discussed in detail how to apply the field-theoretical approach to study ageing phenomena in the bulk of a critical system. Taking advantage of previous works on the subject [145, 147, 148] we outline here a scaling approach to the non-equilibrium dynamics in the presence of spatial surfaces. Let us go back to equations (84) and (85), valid

for a critical system in the bulk, i.e., far enough from any surface. From these equations it is possible to compute the two-point correlation and response functions in real space, which depend on the coordinates of the two points $(\mathbf{x}', t')$ and $(\mathbf{x}, t)$. In the following we assume $t > t'$. Because of the translational invariance in the bulk, this dependence is only on $\mathbf{x}' - \mathbf{x}$. Let us introduce in the system a planar $(d-1)$-dimensional surface $\mathcal{S}$. As a consequence, the translational invariance is broken in the direction perpendicular to $\mathcal{S}$. For later convenience we decompose the position vector as $\mathbf{x} = (\mathbf{x}_\parallel, x_\perp)$ where $\mathbf{x}_\parallel$ is the $(d-1)$-dimensional component parallel to $\mathcal{S}$ and $x_\perp$ is the one-dimensional component perpendicular to it. The residual translational symmetry along $\mathcal{S}$ implies that the two-point correlation functions depend on $\mathbf{x}'_\parallel$ and $\mathbf{x}_\parallel$ through $\mathbf{x}'_\parallel - \mathbf{x}_\parallel$ whereas they depend on $x'_\perp$ and $x_\perp$ separately. The dependence on $x'_\perp - x_\perp$ is recovered only in the bulk, i.e., far away from $\mathcal{S}$. For $\mathbf{q}_\parallel = \mathbf{0}$ (where $\mathbf{q}_\parallel$ is the momentum conjugate to $\mathbf{x}'_\parallel - \mathbf{x}_\parallel$) one can write equations (84) and (85) as

$$R_{\mathbf{q}_\parallel=\mathbf{0}}(t, t'; x_\perp, x'_\perp) = \quad (t-t')^{a-1/z}(t/t')^\theta F_R^{\mathrm{b}}(x_\perp(t-t')^{-1/z}, x'_\perp(t-t')^{-1/z}, t'/t), \tag{140}$$

$$C_{\mathbf{q}_\parallel=\mathbf{0}}(t, t'; x_\perp, x'_\perp) = t'(t-t')^{a-1/z}(t/t')^\theta F_C^{\mathrm{b}}(x_\perp(t-t')^{-1/z}, x'_\perp(t-t')^{-1/z}, t'/t), \tag{141}$$

where $a = (2-\eta-z)/z$. The superscript b reminds that these forms are valid, *a priori*, in the bulk. As discussed in [145] no genuinely new divergences arise when discussing the non-equilibrium dynamics in the presence of a surface. This means that the short-time expansion $t' \to 0$ of the fields $\varphi(\mathbf{x}', t')$ and $\tilde{\varphi}(\mathbf{x}', t')$ (see equations (77) and (78)) is not influenced by the presence of $\mathcal{S}$ and, conversely, that their short-distance expansion for $x'_\perp \to 0$ is not influenced by the presence of the time surface. Accordingly, the SDE of $\varphi(\mathbf{x}', t')$ and $\tilde{\varphi}(\mathbf{x}', t')$ are the same (at least in model A) as in the static case and have a SDE coefficient $C_*(x'_\perp)$ $[\varphi(\mathbf{x}', t') \sim C_*(x'_\perp)\varphi(\mathbf{x}' = (\mathbf{x}'_\parallel, x'_\perp = 0), t')$ and analogous for $\tilde{\varphi}(\mathbf{x}', t')]$ that in the scaling regime is given by [80]

$$C_*(x'_\perp) = a_*[x'_\perp]^{(\beta_1-\beta)/\nu}. \tag{142}$$

The exponent $\beta_1$ is defined analogously to the bulk exponent $\beta$ and describes the behaviour of the surface magnetization $m_1$ upon approaching the critical point. For the $O(N)$ universality class field-theoretical estimates of $\beta_1$ have been provided using different approaches. For recent reviews see [138, 139]. Equation (142) implies that for $x_\perp \to 0$, $R_{\mathbf{q}_\parallel=\mathbf{0}}, C_{\mathbf{q}_\parallel=\mathbf{0}} \sim [x_\perp]^{(\beta_1-\beta)/\nu}$ i.e., they have no analytic behaviour in this limit. The previous scaling forms can be written in a way that singles out this limiting behaviour. As a result the scaling forms in the presence of a surface are given by (hereafter we omit the subscript $\mathbf{q}_\parallel = \mathbf{0}$)

$$R(t, t'; x_\perp, x'_\perp) = A_R^{\mathrm{s}}(t-t')^{a_1}(t/t')^\theta [x_\perp x'_\perp]^{(\beta_1-\beta)/\nu} \times F_R^{\mathrm{s}}(x_\perp(t-t')^{-1/z}, x'_\perp(t-t')^{-1/z}, t'/t), \tag{143}$$

$$C(t, t'; x_\perp, x'_\perp) = A_C^{\mathrm{s}} t'(t-t')^{a_1}(t/t')^\theta [x_\perp x'_\perp]^{(\beta_1-\beta)/\nu} \times F_C^{\mathrm{s}}(x_\perp(t-t')^{-1/z}, x'_\perp(t-t')^{-1/z}, t'/t), \tag{144}$$

where $a_1 = a - 1/z - 2(\beta_1-\beta)/(\nu z)$. The non-universal amplitudes $A_{R,C}^{\mathrm{s}}$ are fixed requiring $F_{R,C}^{\mathrm{s}}(0,0,0) = 1$. The previous equations provide an extension of the scaling forms presented in [145, 148]. As a consequence of equation (144) one gets

$$\partial_{t'} C(t, t'; x_\perp, x'_\perp) = A_{\partial C}^{\mathrm{s}}(t-t')^{a_1}(t/t')^\theta [x_\perp x'_\perp]^{(\beta_1-\beta)/\nu} \times F_{\partial C}^{\mathrm{s}}(x_\perp(t-t')^{-1/z}, x'_\perp(t-t')^{-1/z}, t'/t), \tag{145}$$

where the non-universal amplitude $A_{\partial C}^{\mathrm{s}}$ is fixed requiring $F_{\partial C}^{\mathrm{s}}(0,0,0) = 1$. It is easy to show that $A_{\partial C}^{\mathrm{s}} = (1-\theta)A_C^{\mathrm{s}}$.

It is natural to define the FDR in analogy to what has been done in the bulk:

$$\mathcal{X}^{\rm s}_{\mathbf{q}_\parallel=\mathbf{0}}(t,t';x_\perp,x'_\perp) \equiv \frac{R(t,t';x_\perp,x'_\perp)}{\partial_s C(t,t';x_\perp,x'_\perp)}. \tag{146}$$

Using the previous scaling forms $\mathcal{X}^{\rm s}_{\mathbf{q}_\parallel=\mathbf{0}}$ can be written as

$$\mathcal{X}^{\rm s}_{\mathbf{q}_\parallel=\mathbf{0}}(t,t';x_\perp,x'_\perp) = \frac{A^{\rm s}_R}{A^{\rm s}_{\partial C}} \frac{F^{\rm s}_R(x_\perp(t-t')^{-1/z}, x'_\perp(t-t')^{-1/z}, t'/t)}{F^{\rm s}_{\partial C}(x_\perp(t-t')^{-1/z}, x'_\perp(t-t')^{-1/z}, t'/t)}. \tag{147}$$

This explicitly shows that $\mathcal{X}^{\rm s}_{\mathbf{q}_\parallel=\mathbf{0}}$, just like $\mathcal{X}_{\mathbf{q}=\mathbf{0}}$, is a universal function, being the ratio of two quantities that have the same scaling dimension. The specific form of $\mathcal{X}^{\rm s}_{\mathbf{q}_\parallel=\mathbf{0}}$ is expected to depend on the universality class of the transition at the surface. Moreover, in the long-time limit $t \gg t'$ and sufficiently close to the surface $x_\perp, x'_\perp \ll (t-t')^{1/z}$ the leading asymptotic value $\mathcal{X}^{\rm s,\infty}$ is approached independently of the distance from the surface, i.e.,

$$\mathcal{X}^{\rm s,\infty} \equiv \lim_{t\to\infty,\ \text{fixed}\, t',x_\perp,x'_\perp} \mathcal{X}^{\rm s}_{\mathbf{q}_\parallel=\mathbf{0}}(t,t';x_\perp,x'_\perp) = \frac{A^{\rm s}_R}{A^{\rm s}_{\partial C}} = \frac{A^{\rm s}_R}{(1-\theta)A^{\rm s}_C}. \tag{148}$$

This is a consequence of the fact that at criticality the influence of the surface deeply penetrates into the bulk as time goes by, whereas a bulk-like behaviour can be observed only for $x_\perp, x'_\perp \gg (t-t')^{1/z}$. Following the argument outlined in section 2.4 one can argue that the asymptotic limit of the FDR just introduced is the same as that obtained by considering in equation (146) the surface two-point autoresponse and autocorrelation functions, i.e., quantities in real space. On the other hand, as its bulk counterpart $\mathcal{X}_{\mathbf{q}=\mathbf{0}}$, the expression in the mixed representation $(\mathbf{q}_\parallel, x_\perp)$ is more suited to be computed within the field-theoretical approach and to detect in numerical simulations the ageing behaviour associated with the slow dynamics of the slow mode $\mathbf{q}_\parallel = \mathbf{0}$.

Let us check the scaling relations (143) and (144) within the Gaussian model and compute the associated FDR (147) in the case of OT and SpT. In the presence of the surface, the equilibrium correlation and response function are given by [80]

$$R^{(\rm s,e)}(\mathbf{q}_\parallel, x_\perp, x'_\perp, \Delta t) = R^{(\rm b,e)}(\mathbf{q}_\parallel, x_\perp - x'_\perp, \Delta t) \pm R^{(\rm b,e)}(\mathbf{q}_\parallel, x_\perp + x'_\perp, \Delta t) \tag{149}$$

$$C^{(\rm s,e)}(\mathbf{q}_\parallel, x_\perp, x'_\perp, \Delta t) = C^{(\rm b,e)}(\mathbf{q}_\parallel, x_\perp - x'_\perp, \Delta t) \pm C^{(\rm b,e)}(\mathbf{q}_\parallel, x_\perp + x'_\perp, \Delta t) \tag{150}$$

for $\begin{smallmatrix}\rm SpT\\ \rm OT\end{smallmatrix}$. $R^{(\rm b,e)}(\mathbf{q}_\parallel, x_\perp, \Delta t)$ and $C^{(\rm b,e)}(\mathbf{q}_\parallel, x_\perp, \Delta t)$ are the bulk equilibrium response and correlation functions in the $(\mathbf{q}_\parallel, x_\perp)$-representation and $\Delta t = t - t'$. Let us recall that in momentum space (we set $\Omega = 1$) they read

$$R^{(\rm b,e)}(\mathbf{q}, \Delta t) = \theta(\Delta t)\, {\rm e}^{-(\mathbf{q}^2+r_0)\Delta t}, \tag{151}$$

$$C^{(\rm b,e)}(\mathbf{q}, \Delta t) = \frac{1}{\mathbf{q}^2 + r_0}\, {\rm e}^{-(\mathbf{q}^2+r_0)|\Delta t|} \tag{152}$$

$[\mathbf{q} = (\mathbf{q}_\parallel, q_\perp)]$. Accordingly,

$$\begin{aligned} R^{(\rm b,e)}(\mathbf{q}_\parallel, \Delta x_\perp, \Delta t) &= \int \frac{{\rm d}q_\perp}{2\pi} R^{(\rm b,e)}(\mathbf{q}, \Delta t)\, {\rm e}^{{\rm i}q_\perp \Delta x_\perp} \\ &= \theta(\Delta t)(4\pi\Delta t)^{-1/2} \exp\left\{-\left(\mathbf{q}_\parallel^2 + r_0\right)\Delta t - \frac{(\Delta x_\perp)^2}{4\Delta t}\right\} \end{aligned} \tag{153}$$

$$C^{(\rm b,e)}(\mathbf{q}_\parallel, \Delta x_\perp, \Delta t) = \int_{|\Delta t|}^{\infty} {\rm d}u\, R^{(\rm b,e)}(\mathbf{q}_\parallel, \Delta x_\perp, u). \qquad \text{[FDT]} \tag{154}$$

As reviewed in section 3, the effect of the quench from the high-temperature disordered phase amounts to the introduction of a time surface in the field-theoretical description of the model.

At the fixed point $\tau_0 = \infty$ the boundary condition at the time surface is a Dirichlet one. Accordingly, the correlation function is modified, compared to the time-bulk one, by the introduction of an 'image term' in time, i.e., $C^{(\mathrm{b,ne})}(\mathbf{q}; t, t') = C^{(\mathrm{b,e})}(\mathbf{q}, t - t') - C^{(\mathrm{b,e})}(\mathbf{q}, t + t')$, whereas the response function is not modified: $R^{(\mathrm{b,ne})}(\mathbf{q}; t, t') = R^{(\mathrm{b,e})}(\mathbf{q}, t - t')$. The presence of a spatial surface in the problem does not change the previous considerations. Accordingly, one finds that for the non-equilibrium evolution (ne) after a quench

$$R^{(\mathrm{s,ne})}(\mathbf{q}_\parallel, x_\perp, x'_\perp; t, t') = R^{(\mathrm{s,e})}(\mathbf{q}_\parallel, x_\perp, x'_\perp, t - t'), \tag{155}$$

$$C^{(\mathrm{s,ne})}(\mathbf{q}_\parallel, x_\perp, x'_\perp; t, t') = C^{(\mathrm{s,e})}(\mathbf{q}_\parallel, x_\perp, x'_\perp, t - t') - C^{(\mathrm{s,e})}(\mathbf{q}_\parallel, x_\perp, x'_\perp, t + t'). \tag{156}$$

By means of equations (149), (150), (153) and (154) it is simple to find that $R^{(\mathrm{s,ne})}$ and $C^{(\mathrm{s,ne})}$ have the expected scaling forms (143) and (144) with the proper (mean-field) values of the critical exponents. Moreover, the non-universal amplitudes are easily computed: $A^{\mathrm{s}}_R[\mathrm{SpT}] = \pi^{-1/2}$, $A^{\mathrm{s}}_R[\mathrm{OT}] = (4\pi)^{-1/2}$, $A^{\mathrm{s}}_C[\mathrm{SpT}] = 2\pi^{-1/2}$ and $A^{\mathrm{s}}_C[\mathrm{OT}] = \pi^{-1/2}$. Using equation (148) one concludes that the surface FDR is equal to $\frac{1}{2}$ within the Gaussian theory for both SpT and OT, as it is the case for the Gaussian bulk FDR.

The previous equations can be rewritten in real space. In particular, the FDT together with equation (156) yields

$$\partial_{t'} C^{(\mathrm{s,ne})}(\mathbf{x}, \mathbf{x}'; t, t') = R^{(\mathrm{s,e})}(\mathbf{x}, \mathbf{x}', t - t') + R^{(\mathrm{s,e})}(\mathbf{x}, \mathbf{x}', t + t'). \tag{157}$$

Accordingly, the FDR in real space can be written as $(t > t')$

$$[X^{\mathrm{s}}(\mathbf{x}, \mathbf{x}'; t, t')]^{-1} = 1 + \frac{R^{(\mathrm{s,e})}(\mathbf{x}, \mathbf{x}', t + t')}{R^{(\mathrm{s,e})}(\mathbf{x}, \mathbf{x}', t - t')}, \tag{158}$$

where $R^{(\mathrm{s,e})}(\mathbf{x}, \mathbf{x}', \Delta t)$ is given by the Fourier transform of equation (149). Let us investigate the cross-over between the bulk-like FDR and the surface-like one. For simplicity we consider the case $\mathbf{x} = \mathbf{x}'$. Then it is easy to find that

$$[X^{\mathrm{s}}(\mathbf{x}, \mathbf{x}; t, t')]^{-1} = 1 + \left(\frac{t - t'}{t + t'}\right)^{d/2} \frac{1 \pm \exp\left\{-x_\perp^2 \big/ [t + t']\right\}}{1 \pm \exp\left\{-x_\perp^2 \big/ [t - t']\right\}}. \tag{159}$$

For $x_\perp^2 \gg t + t'$ one recovers the bulk value of the FDR equation (128) both for SpT and OT. In the opposite limit $x_\perp^2 \ll t - t'$ for the OT one finds

$$[X^{\mathrm{s}}(\mathbf{x}, \mathbf{x}; t, t')]^{-1} = 1 + \left(\frac{t - t'}{t + t'}\right)^{d/2+1} \tag{160}$$

that differs from the corresponding bulk quantity and displays the same cross-over from the quasi-equilibrium regime $t \gtrsim t'$ to the ageing regime $t \gg t'$. On the other hand, in the case of SpT, one finds the same expression as in the bulk, given in equation (128). These results for the surface FDR provide the Gaussian theoretical predictions for the quantity $X_1$ numerically investigated in [148]. In particular, as far as the asymptotic value $X_1^\infty$ is concerned, the Gaussian result is $X_1^\infty = \frac{1}{2}$, both for OT and SpT. At first sight, the non-monotonic behaviour of $X_1^\infty[\mathrm{OT}]$ as a function of the space dimensionality, found in [148], can be hardly accounted for within the field-theoretical approach. Further analytical and numerical investigations are required to clarify this issue.

## 6. Dynamics of a conserved order parameter (model B)

In this section, we discuss the effect of the conservation law of the order parameter. As explained in section 3.4 this defines the model B universality class. Compared to the model A,

model B is much less studied in the literature, mainly because, due to the conservation law, its critical slowing down is much more severe and it is quite difficult to reach the asymptotic behaviour in numerical and experimental investigations.

### *6.1. Review of numerical and analytic results*

For model B dynamics, the critical exponent $z$ is $z = 4 - \eta$ exactly [26] and it has been argued that $\lambda = d$ [149], as found in several numerical studies for the one- and two-dimensional Ising models [149–153].

Earlier claims that the one-dimensional FD plot (and hence the FDR) of the Kawasaki–Ising model reduces asymptotically to the Glauber one [52] have been recently revisited in [152] and the apparent equivalence between the two models has been traced back to the time range on which the analysis of [52] was based.

The scaling form of the autocorrelation function has attracted particular interest. For $t, s \gg 1$ it has been found to scale like [152]

$$C_{\mathbf{x}=0}(t, s) \sim t^{-d/z} \mathcal{C}(ts^{-\phi}), \tag{161}$$

where $\mathcal{C}(x)$ approaches a non-zero constant for $x \to \infty$ whereas $\mathcal{C}(x) \sim x^{(d-\lambda')/z}$ for $x \ll 1$ and $\phi = 1 + (2 - \eta)/(\lambda' - d)$. This scaling implies the existence of a new relevant length scale $\sim s^{\phi/z}$ which controls the crossover between the two regimes. Based on MC simulations of the one- and two-dimensional Ising models with Kawasaki dynamics it has been conjectured that $\lambda' = d + 3/2$ [152]. This value has been later revised by Sire, who calculated exactly the correlation function for the $O(\infty)$ model, finding $\lambda' = d + 2$ [153]. By means of simulations of the Ising model in $d = 1, 2$ with an accelerated dynamics (in the same dynamic universality class as model B), the relation $\lambda' = d + 2$ has been checked and it has been conjectured to generally hold for all systems with conserved order parameter.

So far numerical simulations, both for $d = 1$ and 2, have not been able to explore the long-time regime from which $X^\infty$ is determined. However the conclusion $X^\infty_{\text{Kawasaki}} > X^\infty_{\text{Glauber}}$ for the critical FDR of the magnetization seems quite robust, at least in two dimensions [152]. This fact is somehow surprising if one interprets $T/X^\infty$ as an effective temperature. Indeed, one would expect that a slower dynamics should result in a higher $T_{\text{eff}}$.

### *6.2. Field-theoretical approach*

The field-theoretical approach to model B dynamics starts from the effective action (39). The presence of the gradient term (that in momentum space gives a factor $\mathbf{q}^2$) makes the difference with model A dynamics. It leads to an interaction vertex of the form $\mathbf{q}^2 g_0$ that reduces the primitive divergences of all Feynman diagrams so that no new divergences are generated by the dynamics [30, 24]. Hence the critical exponents are exactly obtained on the sole basis of power counting: $\tilde{\eta} = -\eta$ (i.e., $z = 4 - \eta$ [30, 28]) and $\eta_0 = \theta = 0$ [24]. Note that the response function in this case is not given by $\langle \tilde{\varphi}_{-\mathbf{q}}(s) \varphi_{\mathbf{q}}(t) \rangle \equiv G_{\mathbf{q}}(t, s)$, but rather by $R_{\mathbf{q}}(t, s) = \sigma \mathbf{q}^2 G_{\mathbf{q}}(t, s)$ (see equation (32)).

### *6.3. Scaling forms*

The general scaling forms may be obtained solving RG equations with the methods of characteristic functions, as outlined in section 3.5.2 for model A dynamics. In particular equations (72) and (73) for the correlation functions are unchanged and they provide quite stringent predictions since one exactly knows that $\tilde{\eta} = -\eta$ and $\eta_0 = 0$.

The scaling of the autoresponse function at criticality is simply obtained from these equations with $\delta(1,1,0) = d$:

$$R_{\mathbf{x}=0}(t,s) = -\sigma\nabla^2_{\mathbf{x}}\langle\varphi(\mathbf{x},t)\tilde{\varphi}(\mathbf{0},s)\rangle\big|_{\mathbf{x}=0} = \mathcal{A}_R(t-s)^{-(d+2)/z}\mathcal{F}_R(s/t), \quad (162)$$

where, as usual, one singles out the non-universal constant $\mathcal{A}_R$, so that $\mathcal{F}_R(v)$ is a universal function with $\mathcal{F}_R(0) = 1$.

The autocorrelation function at the fixed point $\tau_0^{-1} = 0$ is obtained from $\delta(2,0,0) = d - 2 + \eta = d + 2 - z$:

$$C_{\mathbf{x}=0}(t,s) = \mathcal{A}_C s(t-s)^{-(d+2)/z}\mathcal{F}_C(s/t), \quad (163)$$

where the proportionality to $s$ is explicitly shown to stress the fact that the Dirichlet boundary condition in time enforces $C_{\mathbf{x}}(t,0) = 0$ for $\tau_0^{-1} = 0$. As emphasized in section 3.6, the long-time limit of the correlation function for finite $s$ can be influenced by the non-vanishing initial correlations, i.e., by a finite $\tau_0$. From equation (94) we get, in the long-time limit and neglecting further corrections coming from $\tau_0^{-1}$

$$A(t) \equiv C_{\mathbf{x}=0}(t,0) = \tau_0^{-1}\langle\varphi(\mathbf{x},t)\tilde{\varphi}_0(\mathbf{x})\rangle \simeq \tau_0^{-1}t^{-\delta(1,0,1)/z} = \tau_0^{-1}t^{-d/z}, \quad (164)$$

where we used $\delta(1,0,1) = d$ from equation (73). Accordingly, for $t \gg s$, $C_{\mathbf{x}=0}(t,0) \simeq \tau_0^{-1}t^{-d/z}$, whereas, from equation (163), $C_{\mathbf{x}=0}(t,s) \simeq st^{-(d+2)/z}$. As a fundamental difference with model A dynamics, these two terms do not scale in the same way with $t$ for $t \gg s$ and in particular the former is the leading one for large $t$. Accordingly the autocorrelation exponent (see equation (96)) is given by $\lambda = d$, in agreement with what has been argued in [149]. Furthermore a cross-over behaviour is expected for finite $s$ when the contribution coming from finite initial correlations is of the same order of magnitude as the contribution with $\tau_0^{-1} = 0$. Comparing the previously given asymptotic behaviours one concludes that the former dominates when $\tau_0^{-1}t^{-d/z} \gg st^{-(d+2)/z}$, i.e., when $\tau_0 st^{-2/z} \ll 1$. This agrees with the prediction (161), with $\phi = z/2$ (equivalently $\lambda' = d + 2$), providing a FT proof of Sire's conjecture [153].

From the scaling of the autoresponse function (see the definition in equation (98)) we instead get $\lambda_R = d + 2 \neq \lambda$. However, such inequality does not prevent the definition of a universal FDR, as one might naively think on the basis of the fact that this leads to a different scaling dimensions for $R$ and $\partial_s C$. Indeed the contribution to $\partial_s C$ coming from finite $\tau_0$, leading in the large-$t$ limit of $C_{\mathbf{x}=0} \simeq \tau_0^{-1}t^{-d/z}$, is expected to scale as $\tau_0^{-1}t^{-d/z-1}$, turning out to be subleading in $\partial_s C_{\mathbf{x}=0}(t,s) \simeq t^{-(d+2)/z}$. In fact the derivative is

$$\partial_s C_{\mathbf{x}=0}(t,s) = \mathcal{A}_{\partial C}(t-s)^{-(d+2)/z}\mathcal{F}_{\partial C}(s/t), \quad (165)$$

where, being $\theta = 0$, $\mathcal{A}_{\partial C} = \mathcal{A}_C$. It is possible to define the universal amplitude ratio

$$X^{\infty} = \lim_{s\to\infty}\lim_{t\to\infty}\frac{R_{\mathbf{x}=0}(t,s)}{\partial_s C_{\mathbf{x}=0}(t,s)} = \frac{\mathcal{A}_R}{\mathcal{A}_{\partial C}} = \frac{\mathcal{A}_R}{\mathcal{A}_C}. \quad (166)$$

### *6.4. Gaussian results*

The Gaussian response and correlation functions read for $t > s$

$$R^G_{\mathbf{q}}(t,s) = \sigma\mathbf{q}^2\langle\tilde{\varphi}(\mathbf{q},s)\varphi(-\mathbf{q},t)\rangle_G = \sigma\mathbf{q}^2\,\mathrm{e}^{-\sigma\mathbf{q}^2(\mathbf{q}^2+r_0)(t-s)}, \quad (167)$$

$$C^G_{\mathbf{q}}(t,s) = \langle\varphi(\mathbf{q},s)\varphi(-\mathbf{q},t)\rangle_G = \frac{\mathrm{e}^{-\sigma\mathbf{q}^2(\mathbf{q}^2+r_0)(t-s)} - \mathrm{e}^{-\sigma\mathbf{q}^2(\mathbf{q}^2+r_0)(t+s)}}{\mathbf{q}^2 + r_0} + \tau_0^{-1}\,\mathrm{e}^{-\sigma\mathbf{q}^2(\mathbf{q}^2+r_0)(t+s)}, \quad (168)$$

where we keep the correction coming from non-zero $\tau_0^{-1}$. The Fourier transforms of the critical functions for $\mathbf{x} = \mathbf{0}$ are (setting $\sigma = 1$)

$$R_{\mathbf{x}=0}(t,s) = r_{1,d}(t-s)^{-(d+2)/4}, \tag{169}$$

$$C_{\mathbf{x}=0}(t,s) = r_{-1,d}s(t-s)^{-(d+2)/4} f_G\left(\frac{s}{t}\right) + \tau_0^{-1} r_{0,d}(t+s)^{-d/4}, \tag{170}$$

where $r_{n,d} = \Gamma((d+2n)/4)/[2(4\pi^{d/2})\Gamma(d/2)]$ and

$$f_G(x) = (1-x)^{(d+2)/4} \frac{(1-x)^{(2-d)/4} - (1+x)^{(2-d)/4}}{x}, \tag{171}$$

which is regular for small $x$: $f_G(0) = (2-d)/2$. These forms agree with those obtained from the solution of the RG equations (see equations (91) and (92)) with $z = 4$.

From the scaling derived in the previous subsection, we can define the universal FDRs in real and reciprocal space as in equations (15) and (18). The argument presented in section 2.4 is easily extended to the present case, concluding $X^{\infty}_{\mathbf{x}=0} = \mathcal{X}^{\infty}_{\mathbf{q}\to\mathbf{0}}$. Here we use the notation $\mathcal{X}^{\infty}_{\mathbf{q}\to\mathbf{0}}$ given that $\varphi_{\mathbf{q}=\mathbf{0}}$ does not change in time, being a conserved quantity. Accordingly, $R_{\mathbf{q}=\mathbf{0}}$ and $\partial_s C_{\mathbf{q}=\mathbf{0}}$ vanish whereas the limit $\mathbf{q} \to \mathbf{0}$ of their ratio is well defined and finite. On the other hand, in a finite system of linear dimension $L$, the smallest allowed mode is $\mathbf{q}_{\min}$ with $|\mathbf{q}_{\min}| \propto 1/L$. The limit $\mathbf{q} \to \mathbf{0}$ is explored by studying the mode with $\mathbf{q} = \mathbf{q}_{\min}$ in the limit of large $L$. In any case to stay within the non-equilibrium regime one has to always consider times much smaller than $L^z$, otherwise the system equilibrates.

The Gaussian FDR can be readily computed using the previous expressions, finding

$$\mathcal{X}^{-1}_{\mathbf{q}}(t,s) = 1 + \left(1 - \frac{\mathbf{q}^2 + r_0}{\tau_0}\right) \mathrm{e}^{-2\sigma\mathbf{q}^2(\mathbf{q}^2+r_0)s}. \tag{172}$$

As for model A, all the modes with $\mathbf{q} \neq \mathbf{0}$ relax to equilibrium, i.e., $\mathcal{X}^{\infty}_{\mathbf{q}\neq 0} = 1$. At variance with the case of model A, $\mathcal{X}_{\mathbf{q}\to\mathbf{0}}(t,s)$ reaches a non-trivial (but not universal) value $1\big/\big(2 + r_0\tau_0^{-1}\big)$ also for $r_0 \neq 0$ (non-critical case). The conservation law allows arbitrarily slow modes which in the case of a purely dissipative dynamics is present only at criticality. The universal critical zero-momentum FDR is again $\frac{1}{2}$.

Since it was instructive for the model A dynamics, let us have a look at the Gaussian FDR in real space. From the Gaussian response and correlation functions reported above one gets

$$X^{-1}_{\mathbf{x}=\mathbf{0}}(t,s) = 1 + \left(\frac{t-s}{t+s}\right)^{(d+2)/4} - \tau_0^{-1} B_d \left(\frac{t-s}{t+s}\right)^{(d+2)/4} (t+s)^{-1/2}, \tag{173}$$

where $B_d = \Gamma(d/4+1)/\Gamma(d/4+1/2)$. The term proportional to $\tau_0^{-1}$ is only a correction to the asymptotic form for $t > s \to \infty$. Anyway, as in the case of model A, $X^{-1}_{\mathbf{x}=\mathbf{0}}$ has a slower approach to the asymptotic value $X^{\infty} = 1/2$ than that displayed by $\mathcal{X}_{\mathbf{q}\to 0}$. This faster approach suggests the use of coherent observables to determine $X^{\infty}$ also in the case with conserved order parameter. Although this involves the limiting procedure described above, which is expected to be more noisy than working in real space, we think that the absence of a long transient allows for a more efficient determination of $X^{\infty}$.

### *6.5. Exact solution for $N = \infty$*

The next step in understanding general properties of $X^{\infty}$ is the exact calculation for $N = \infty$. Some quantities were already considered in the literature [84, 153]. In this limit the interaction can be self-consistently decoupled as $g_0\varphi^4(\mathbf{x},t) \to g/N C_{\mathbf{x}=\mathbf{0}}(t,t)\varphi^2(\mathbf{x},t)$ as for model A dynamics. At criticality this leads to the response function $R_{\mathbf{q}}(t,s) = \mathbf{q}^2 G_{\mathbf{q}}(t,s)$ where

$$G_{\mathbf{q}}(t,s) = \mathrm{e}^{-\mathbf{q}^4(t-s)}\, \mathrm{e}^{2a_d\mathbf{q}^2(\sqrt{t}-\sqrt{s})}. \tag{174}$$

The constant $a_d$ is determined by a consistency condition (see [84], section IV). Hence, from the definition of the correlation function, one gets [153]

$$C_{\mathbf{q}}(t,s) = 2\mathbf{q}^2 \int_0^s \mathrm{d}u\, G_{\mathbf{q}}(t,u) G_{\mathbf{q}}(s,u) + \tau_0^{-1} G_{\mathbf{q}}(t,0) G_{\mathbf{q}}(s,0), \tag{175}$$

explicitly showing that all the exponents are the mean-field ones, as expected. The FDR $\mathcal{X}_{\mathbf{q}\to\mathbf{0}}(t,s)$ turns out to be $1/2$ for all times and without corrections, as an important difference with model A dynamics, where $X^\infty = 1 - 2/d$ for $2 \leqslant d \leqslant 4$ and $N = \infty$.

### *6.6. FDR beyond the Gaussian approximation*

It is easy to realize that the one-loop diagrams do not contribute to the critical FDR for $\mathbf{q} \to \mathbf{0}$. In fact, as in model A, they are given by tadpoles [19]. Their one-particle irreducible parts are independent of the external momentum $\mathbf{q}$. On the other hand, the vertex carries a factor $\mathbf{q}^2$. Therefore, in the limit $\mathbf{q}^2 \to 0$, the contributions of these diagrams vanish. Possibly non-vanishing corrections (if any) could come from diagrams whose one-particle irreducible part depends on $\mathbf{q}$ (as the two-loop sunset diagram). So one concludes that

$$X^\infty = \tfrac{1}{2} + O(\epsilon^2). \tag{176}$$

A more detailed analysis of this problem is in progress [154].

Let us compare this result for $X^\infty_{\text{Kawasaki}}$ with that one for model A $\left(X^\infty_{\text{Glauber}}\right)$. By using equation (132) one finds that $X^\infty_{\text{Kawasaki}} \big/ X^\infty_{\text{Glauber}} = 1 + (N+2)/[4(N+8)]\epsilon + O(\epsilon^2)$. Therefore, for small $\epsilon > 0$ one has $X^\infty_{\text{Kawasaki}} > X^\infty_{\text{Glauber}}$ a relation that should not be changed by higher-order terms up to rather large $\epsilon$. This expectation is in agreement with the numerical results of [152], where such a relation has been observed for the two-dimensional Ising model.

## 7. Coupling of a conserved density to the non-conserved order parameter (model C)

We now address the problem of when and how the critical behaviour is influenced by coupling a non-conserved order parameter to a conserved density. As discussed in section 3.4, such a coupling characterizes the so-called model C universality class. The starting point is the effective dynamic action (45) defined in terms of the two fields $\varphi$ and $\varepsilon$. The propagators (Gaussian two point correlation and response functions) of the resulting theory are the same as model A (equations (112) and (113)) as far as the order parameter (and the associated response field) are concerned, whereas, for $\varepsilon$ and $\tilde{\varepsilon}$ one has [81]

$$\langle \tilde{\varepsilon}(\mathbf{q},s)\varepsilon(-\mathbf{q},t)\rangle_G = R^G_{\varepsilon,q}(t,s) = \theta(t-s) G_\varepsilon(t-s), \tag{177}$$

$$\langle \varepsilon(\mathbf{q},s)\varepsilon(-\mathbf{q},t)\rangle_G = C^G_{\varepsilon,q}(t,s) = G_\varepsilon(|t-s|) + (c_0 - 1) G_\varepsilon(t+s), \tag{178}$$

with $G_\varepsilon(t) = \mathrm{e}^{-\rho\Omega q^2 t}$.

Let us briefly recall the scenario of fixed points for non-equilibrium model C [155, 156, 81]. The fixed-point values for the couplings $g$ and $\gamma$ are determined only by the statics. One has $\tilde{g}^* = \tilde{g}^*_A + 6\tilde{\gamma}^{2*}$, where $\tilde{g}^*_A = 6\epsilon/(N+8) + O(\epsilon^2)$ is the fixed-point value of the coupling constant for model A [30].

The value of $\gamma$ at the stable infrared fixed point depends on the sign of the specific-heat exponent $\alpha$:

$$\tilde{\gamma}^{2*} = \begin{cases} 0, & \text{stable for } \alpha < 0, \quad \text{case I}, \\ \dfrac{4-N}{N(N+8)}\epsilon + O(\epsilon^2), & \text{stable for } \alpha > 0, \quad \text{case II}, \end{cases} \tag{179}$$

in the case I, the dynamics of the conserved density decouples from that one of the order parameter and one gets back to model A (at least asymptotically). At the leading order in $\epsilon$-expansion one has, for the $O(N)$ model [30],

$$\alpha = \frac{4-N}{2(N+8)}\epsilon + O(\epsilon^2), \tag{180}$$

therefore the truly model C dynamics fixed point is stable for $N < 4 + O(\epsilon)$. In three dimensions, theoretical and experimental investigations showed that $\alpha$ is negative for $N \geqslant 2$ (see [67] for a comprehensive review). As a consequence the model C dynamics may be realized only for the three-dimensional Ising model ($N = 1$) which has a positive $\alpha$ [67]. In two dimensions, the boundary value between positive and negative $\alpha$ is $N = 1$, so no physically relevant genuine model C dynamics exists in this case. Albeit for $d = 3$ with $N \geqslant 2$ and $d = 2$ with arbitrary $N$ the dynamics is asymptotically model A, a strong crossover is expected, especially in models with small $\alpha$ [156].

As far as $\rho$ is concerned we have two possible stable fixed points determined by the equilibrium dynamics [156]

(a) $\rho^* = \infty$, stable for $N > N_1(\epsilon) = 4 - [15/4 + 3/2\log(4/3)]\epsilon + O(\epsilon^2)$;
(b) $\rho^* = 2/N - 1 + O(\epsilon)$, stable for $N < N_1(\epsilon)$.

A third region with $\rho^* = 0$, found in earlier calculations [155], has been recently proved to be only an artefact of the $\epsilon$-expansion and consequently it does not exist in any dimension [156]. This (fake) fixed point with $\rho^* = 0$ has been mentioned and discussed in many papers so far [22, 26, 30, 81]. Any reference to it can be ignored.

Finally, as far as the non-equilibrium dynamics is concerned, it has been shown that, whenever $\alpha > 0$, the fixed-point value for $c$ is $c^* = 0$ [81].

We focus here our attention on the only relevant stable fixed point of the model, i.e., II (b). At this fixed point $z = 2 + \alpha/\nu$ exactly [30], so that $z$ can be obtained from the precise estimates of static exponents [67]. $\theta$ instead is an independent exponent known up to two loops [81] in $\epsilon$ expansion. In [22], we obtained the one-loop non-equilibrium universal scaling function of the response

$$F_R(v) = 1 + \frac{4-N}{4(N+8)(N-1)}\ln[(1+(N-1)v)^{N-2}(1-(N-1)v)^N]\epsilon + O(\epsilon^2), \tag{181}$$

that renders, for the physically relevant case of $N = 1$,

$$F_R(v) = 1 - \epsilon\frac{v}{6} + O(\epsilon^2). \tag{182}$$

Also the scaling function $F_C(v)$ has been computed [22] but we do not report it here. For both the universal functions corrections to the Gaussian value already at one-loop order have been found, at variance with model A.

In [22], the FDR has been computed, finding

$$X^\infty = \frac{1}{2}\left\{1 + \frac{4-N}{N+8}\epsilon\left[\frac{N-1}{(4-N)(2-N)} + \frac{N(2-N)}{4(N-1)^2}\ln[N(2-N)]\right]\right\} + O(\epsilon^2), \tag{183}$$

that, for $N = 1$, is exactly the same as in model A, $X^\infty = 1/2(1-\epsilon/12)+O(\epsilon^2)$. This is really surprising, in particular looking at the complicated expression for general $N$. It is probably a coincidence of one-loop computation, but only higher-loop calculations may clarify whether this is a deeper property or not.

## 8. Model A dynamics of a weakly dilute Ising model

A question of theoretical and experimental interest is whether and how the critical behaviour is altered by introducing in the systems a small amount of uncorrelated non-magnetic impurities leading to models with quenched disorder.

The static critical behaviour of these systems with bond or site disorder (the effect of a random field is more complicated see, e.g., [157]) is well understood thanks to the Harris criterion [158]. It states that the addition of impurities to a system which undergoes a second-order phase transition does not change the critical behaviour if the specific-heat critical exponent $\alpha_{\rm p}$ of the pure system is negative. If $\alpha_{\rm p}$ is positive, the transition is altered.

For the very important class of the three-dimensional $O(N)$-vector models, it is known that $\alpha_{\rm p} < 0$ for $N \geqslant 2$ [67] and the critical behaviour is unchanged in the presence of weak quenched disorder (apart from large crossover effects, see, e.g., [159, 160]). Instead, the specific-heat exponent of the three-dimensional Ising model is positive [67]. Therefore, the introduction of a small amount of non-magnetic impurities (dilution) leads to a new universality class (as confirmed by RG analyses, Monte Carlo simulations and experiments, see [67, 161, 162] as comprehensive reviews on the subject) to which the weakly dilute random Ising model (RIM) belongs. The RIM is a lattice spin model with nearest-neighbour interaction Hamiltonian

$$\mathcal{H}_c = -\sum_{\langle ij \rangle} \rho_i \rho_j s_i s_j, \tag{184}$$

where $s_i$ are the Ising spins at the site $i$, and $\rho_i$ are uncorrelated quenched random variables such that $\rho_i = 1$ with probability $c$ ($0 < c \leqslant 1$ being the spin concentration in the lattice), $\rho_i = 0$ with probability $1 - c$. Above the percolation threshold $c^*$, the critical properties of the model (184) are predicted to be independent of the actual impurity concentration [67, 161, 162]. However to make this property apparent, the corrections to the scaling have to be properly taken into account in the analysis of numerical and experimental data (see, e.g., [67, 159, 160, 163]).

The purely relaxational equilibrium dynamics (model A) of this new universality class is qualitatively well understood [109, 164–169]. The dynamic critical exponent $z$ differs from the mean-field value already in the one-loop approximation [164], at variance with the pure model. This exponent is known up to two loops in a $\sqrt{\epsilon}$ [166] and up to three loops in fixed-dimension expansion [167] in three dimensions. Unfortunately the agreement between FT estimates (giving $z \simeq 2.2$), numerical [169, 168, 61] ($z \simeq 2.6$) and experimental [170] ($z \simeq 1.9 - 2.2$) estimates is still quite poor.

The non-equilibrium dynamics is instead less studied. Within the $\sqrt{\epsilon}$ expansion, $\theta$ was determined up to two-loop order [85] and the response function up to one loop, both for conserved and non-conserved order parameter [84]. A careful three-dimensional numerical simulation [61], taking properly into account the corrections to the scaling, showed that even $\theta$ is dilution independent and its value is (quite surprisingly, compared with $z$) in good agreement with the $\sqrt{\epsilon}$ expansion. In fact the obtained value $\lambda/z = 1.05(3)$ [61], using the scaling law $\lambda/z = d - \theta'$, leads to $\theta' = 0.100(35)$ that has to be compared with the two-loop FT result $\theta'_{\rm 2loops} \simeq 0.087$ [85]. Also the autocorrelation scaling function obtained by means of numerical simulation [61] is in nice agreement with the RG prediction (92). The finite-size scaling in the non-equilibrium regime has also been investigated [171].

The time evolution of the weakly dilute Ising model is described by the stochastic Langevin equation (33) with $\mathcal{H} \mapsto \mathcal{H}_\psi$, where the static Landau–Ginzburg Hamiltonian $\mathcal{H}_\psi[\varphi]$ has a

space-dependent random temperature [172]

$$\mathcal{H}_\psi[\varphi] = \int \mathrm{d}^d x \left[ \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}(r_0 + \psi(\mathbf{x}))\varphi^2 + \frac{1}{4!} g_0 \varphi^4 \right]. \tag{185}$$

Here $\psi(\mathbf{x})$ is a spatially uncorrelated random field with Gaussian distribution

$$P(\psi) = \frac{1}{\sqrt{4\pi w}} \exp\left[ -\frac{\psi^2}{4w} \right]. \tag{186}$$

Dynamical correlation functions, generated by Langevin equation and averaged over the noise $\zeta$, are given by equation (35) with $\mathcal{H}$ replaced by $\mathcal{H}_\psi$ (we denote by $S_\psi$ the resulting dynamical functional).

In the analysis of static critical behaviour, the average over the quenched disorder $\psi$ is usually performed by means of the replica trick [173, 172]. If instead one is interested in dynamic processes it is simpler to perform directly the average at the beginning of the calculation [174]

$$\int [\mathrm{d}\psi] P(\psi) \exp(-S_\psi[\varphi, \tilde{\varphi}]) = \exp(-S[\varphi, \tilde{\varphi}]) \tag{187}$$

obtaining the $\psi$-independent action [85] (with $v_0 \propto w$)

$$S[\varphi, \tilde{\varphi}] = \int \mathrm{d}^d x \left\{ \int_0^\infty \mathrm{d}t\, \tilde{\varphi} \left[ \partial_t \varphi + \Omega(r_0 - \Delta)\varphi - \Omega\tilde{\varphi} \right] + \frac{\Omega g_0}{3!} \int_0^\infty \mathrm{d}t\, \tilde{\varphi}\varphi^3 - \frac{\Omega^2 v_0}{2} \left( \int_0^\infty \mathrm{d}t\, \tilde{\varphi}\varphi \right)^2 \right\}. \tag{188}$$

Note that a non-local (in time) effective interaction term has been generated.

The perturbative expansion is performed in terms of the two fourth-order couplings $g_0$ and $v_0$ and using the propagators given in equations (112) and (113). The scaling of the response function (see equation (86)) is characterized by [21]

$$A_R = 1 - \frac{1}{2}\sqrt{\frac{6\epsilon}{53}}\gamma_E + O(\epsilon), \qquad \text{and} \qquad F_R(x) = 1 + O(\epsilon), \tag{189}$$

and the correlation (see equation (87)) by [21]

$$\frac{A_C}{2} = 1 + \frac{1}{2}\sqrt{\frac{6\epsilon}{53}}(2 - \gamma_E) + O(\epsilon), \tag{190}$$

$$F_C(x) = 1 + \frac{1}{2}\sqrt{\frac{6\epsilon}{53}} \left[ 1 + \frac{1}{2} \left( 1 + \frac{1}{x} \right) \log \frac{1-x}{1+x} \right] + O(\epsilon). \tag{191}$$

Note that, at variance with the pure model, the function $F_C(x)$ gets a contribution already at one-loop order. $F_R(v)$ remains 1 in apparent agreement with LSI although $z \neq 2$, already at this order.

The FDR for generic times is

$$\mathcal{X}_{\mathbf{q}=\mathbf{0}}(s/t) = \frac{1}{2} \left( 1 - \frac{1}{2}\sqrt{\frac{6\epsilon}{53}} \left[ 1 + \frac{1}{2} \log \frac{1 - s/t}{1 + s/t} \right] \right) + O(\epsilon). \tag{192}$$

Therefore, assuming even in this case the validity of equation (21), one finds

$$X^\infty = \frac{1}{2} - \frac{1}{4}\sqrt{\frac{6\epsilon}{53}} + O(\epsilon), \tag{193}$$

that for $\epsilon = 1$ (that is the only physical relevant case, given that $\alpha_{\rm p} = 0$ in the two-dimensional Ising model) leads to $X^\infty \simeq 0.416$. To this order it is not clear whether randomness really changes $X^\infty$ in a sensible way or not. In any case this could not be safely stated from low-order computations since the $\sqrt{\epsilon}$ is known to be not well-behaved at $d = 3$ [67, 161]. However, a qualitative conclusion can be drawn from equation (192): $\mathcal{X}_{\mathbf{q}=\mathbf{0}}$ has a quite strong dependence on the time ratio $s/t$, that, at variance with the pure model, should be easily identified in simulations.

The relation between $X^\infty_{\mathbf{x}=0}$ and $\mathcal{X}^\infty_{\mathbf{q}=0}$ in the RIM has been studied in more detail by Schehr and Paul [61]. Due to the quench disorder, the response function $R_{\mathbf{q}}$ has a power-law decay for $q^z t \gg 1$ [84]. Therefore the argument leading to $X^\infty_{\mathbf{x}=0} = \mathcal{X}^\infty_{\mathbf{q}=0}$ does not apply straightforwardly in the form outlined in section 2.4. To clarify this point $R_{\mathbf{q}}(t, s)$ and $C_{\mathbf{q}}(t, s)$ were calculated for generic momenta up to the first order in the $\sqrt{\epsilon}$-expansion. Then, via the Fourier transform, one can compute explicitly $R_{\mathbf{x}=0}(t, s)$ and $C_{\mathbf{x}=0}(t, s)$ and therefore $X_{\mathbf{x}=0}(t, s)$. In doing that one faces the problem that $R_{\mathbf{q}}$ and $C_{\mathbf{q}}$ [61] have a power-law decay for large $q$ and fixed times, in contrast to the exponential one of the pure model. As a consequence one has to introduce an ultraviolet cut-off $\Lambda_0$, finding [61]:

$$R_{\mathbf{x}=0}(t, s) = \left[A^0_R + A^1_R \log(t - s)\right](t - s)^{a-d/z} \left(\frac{t}{s}\right)^\theta, \tag{194}$$

$$C_{\mathbf{x}=0}(t, s) = \left[A^0_C + A^1_C \log(t - s)\right](t - s)^{a+1-d/z} \left(\frac{t}{s}\right)^\theta F_C(s/t), \tag{195}$$

where $A^{0,1}_{R,C}$ are non-universal amplitudes (with $A^1_R = A^1_C$). In particular $A^0_{R,C}$ explicitly depend on $\Lambda_0$ and are not finite for $\Lambda_0 \to \infty$. Note that these expressions do *not* agree with the general RG predictions equations (91) and (92), because of the presence of the logarithmic term. However this fact is not surprising: logarithmic corrections to the scaling behaviour are generically expected in the presence of quenched disorder, as shown by Cardy using conformal invariance in generic dimension [175].

We point out (as already noted in [61]) that, up to one-loop order, the logarithms can be absorbed in the exponent of $(t - s)$, giving rise to non-universal, cut-off-dependent exponents $a^{(\rm eff)}_{C,R} = a + A^1_{C,R}/A^0_{C,R}$. This possibility seems rather unlikely, but only a two-loop calculation can rule it out. We mention that the same kind of logarithmic behaviour has been found in pinned elastic interfaces near the depinning transition [176].

From equations (194) and (195), one easily derives the FDR in the real space [61]

$$X^{-1}_{\mathbf{x}=0} = F_X(t/s); \qquad \text{with} \quad F_X(u) = 2\frac{u^2 + 1}{(u + 1)^2} + \sqrt{\frac{6\epsilon}{53}} \left(\frac{u - 1}{u + 1}\right)^2 + O(\epsilon), \tag{196}$$

where $F_X(u)$ interpolates between 1 in the quasi-equilibrium regime for $u \to 1$, and its asymptotic value for $u \to \infty$ given by $\lim_{u\to\infty} F_X(u)^{-1} = X^\infty$ of equation (193). This result explicitly shows, at order $\sqrt{\epsilon}$, that the asymptotic FDR for total and local magnetization are indeed equal, despite the power-law behaviour of $R_{\mathbf{q}}$ and $C_{\mathbf{q}}$ for large $q$. This fact calls for a deeper understanding of the relation between $X^\infty_{\mathbf{x}=0}$ and $\mathcal{X}^\infty_{\mathbf{q}=0}$ and of the fact that equation (21) seems to hold independently of the argument presented in section 2.4.

## 9. Purely dissipative dynamics of a $\varphi^3$ Landau–Ginzburg theory

So far we have only considered models whose static critical properties are described by the effective $\varphi^4$ Landau–Ginzburg Hamiltonian given in equation (22). In all these cases $\theta \geqslant 0$ and $0 \leqslant X^\infty \leqslant 1/2$. To understand whether these bounds on such universal quantities are

general properties or they are accidentally connected with the diagrammatic properties of $\varphi^4$ theories, it is worth studying different effective Hamiltonians. In particular, we consider a model whose static critical properties are described by the $\varphi^3$ Landau–Ginzburg Hamiltonian

$$\mathcal{H}_3[\varphi] = \int \mathrm{d}^d x \left[ \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2} r_0 \varphi^2 + \frac{g_0}{3!} \sum_{i,j,k=1}^{N} d_{ijk}\varphi_i\varphi_j\varphi_k \right], \tag{197}$$

where the coefficients $d_{ijk}$ specify the model and $\varphi$ is a $N$-component field. A variety of critical phenomena belongs to the universality class of equation (197). Examples are provided by the isotropic-to-nematic phase transition in liquid crystals [177], the $N+1$-state Potts model [178], its percolation transition ($q = N + 1 \to 1$) [179] and its limit $q \to 0$, representing electrical resistor networks [179]. The model with one component $N = 1$ and $d_{111} = i$ describes the Yang–Lee edge singularity [180]. Moreover a $\varphi^3$-like interaction is one of the most relevant in the effective field-theoretical action of the Edwards–Anderson spin glass [181].

The upper critical dimension for the universality class specified by equation (197) is $d_c = 6$. Therefore it is possible to compute critical exponents and all other universal quantities in a $\epsilon$-expansion with $\epsilon = 6 - d$ [182–185]. Fixed-dimension results have also been provided [186].

However, the field-theoretical study of the Hamiltonian (197) presents several potential pitfalls. For example in the $\epsilon$-expansion of the Potts-symmetric Hamiltonian, a stable fixed point of the renormalized coupling constant $g$ is found only for $N < 7/3$ and its value $g^*$ diverges as $N$ approaches $7/3$, whereas it takes imaginary values for $N > 7/3$. On the other hand, the $\epsilon$ expansion is not reliable for all values of $N$ for which $g^*$ is not sufficiently small [184]. This condition naively leads to the conclusion that the $\epsilon$-expansion makes sense only for $N \leqslant 1$. In spite of these problems the universality class of the percolation transition ($N = 0$) has to be considered free from pitfalls. This fact is corroborated by the remarkable agreement of field-theoretical estimates with numerical and experimental results [187] in $d > 2$ (in $d = 2$ it was claimed that the $\epsilon$ expansion breaks down due to the relevance of $\varphi^4$ interactions [188]).

The simplest dynamics that one can think of realizing for some of the model just described is the purely dissipative one, i.e., the model A of section 3.4, defined by the Langevin equation (33). The dynamic critical properties can be worked out from the dynamical functional (35) with $\mathcal{H}$ replaced by $\mathcal{H}_3$. We point out that the dynamics just introduced is *not* the actual dynamics of isotropic percolation (reviewed, e.g., in [189]).

The calculation of the one-loop non-equilibrium response and correlation function following a quench from the disordered state (introduced by means of $\mathcal{H}_0$, as in section 3.5.2) to the critical point is straightforward and we do not provide any details but only the final results. Zero-momentum response and correlation functions satisfy the expected scaling laws (86) and (87) with exponents

$$a = -\frac{\alpha}{4(\alpha - 4\beta)}\epsilon + O(\epsilon^2), \qquad \theta = -\frac{2\alpha - \gamma}{8(\alpha - 4\beta)}\epsilon + O(\epsilon^2), \tag{198}$$

($a$ agrees with the known exponents [184, 190] through the scaling relation $a = (2 - \eta - z)/z$) and scaling functions

$$F_R(x) = 1 - \epsilon \frac{\alpha}{4(\alpha - 4\beta)} x + O(\epsilon^2), \tag{199}$$

$$F_C(x) = 1 - \epsilon \frac{\alpha}{8(\alpha - 4\beta)} \left[ x - \left(1 + \frac{1}{x}\right) \log(1 - x^2) + O(\epsilon^2) \right]. \tag{200}$$

The resulting FDR is given by

$$X^\infty = \frac{1}{2}\left[ 1 + \frac{3\alpha - \gamma}{8(\alpha - 4\beta)}\epsilon \right] + O(\epsilon^2). \tag{201}$$

In the previous equations we introduced the three contractions

$$d_{ijk}d_{jkl} = \alpha\delta_{il}, \qquad d_{ilm}d_{jmn}d_{knl} = \beta d_{ijk}, \qquad d_{ijk}d_{kll} = \gamma\delta_{ij}, \tag{202}$$

that completely characterize the model at one-loop level. Accordingly, for the models we are interested in, we do not report the coefficients $d_{ijk}$ [178, 180], but only the contractions:

Yang–Lee edge singularity: $\alpha = -1, \qquad \beta = -1, \qquad \gamma = -1,$ (203)

$(N+1)$-state Potts Model: $\alpha = (N+1)^2(N-1), \qquad \beta = (N+1)^2(N-2), \qquad \gamma = 0.$ (204)

The obtained $F_R(x)$ disagrees at one-loop level with the prediction of local scale invariance $F_R(x) = 1$ [135]. This fact is not surprising, since at one-loop level, we have $z \neq 2$ (see the remarks in section 4.3).

Let us discuss these results for some specific model. For the Yang–Lee edge singularity we found

$$\theta = \frac{1}{56}\epsilon + O(\epsilon^2), \qquad X^\infty = \frac{1}{2}\left[1 - \frac{1}{28}\epsilon + O(\epsilon^2)\right], \tag{205}$$

i.e., $\theta > 0$ and $X^\infty < 1/2$. For the models with the symmetry of the $q = (N+1)$-state Potts model, instead

$$\theta = \frac{1-N}{8(7-3N)}\epsilon + O(\epsilon^2), \qquad X^\infty = \frac{1}{2}\left[1 - \frac{3}{8}\frac{1-N}{7-3N}\epsilon + O(\epsilon^2)\right]. \tag{206}$$

For $N < 1$, where the perturbative expansion of such models is considered under control, we get $\theta > 0$ and $X^\infty < 1/2$. Conversely, for $N > 1$ we apparently get $X^\infty > 1/2$ and $\theta < 0$. However, such a result is not completely under control, given that one finds $\nu < 1/2$ [184] for $N > 1$, in disagreement with the results for the $q = (N+1)$-state Potts model [191].

## 10. Conclusions and perspectives

We reviewed the ageing properties of systems which are quenched from the high-temperature phase exactly to their critical points. It clearly emerges that the theoretical understanding of such phenomena is quite complete in comparison with the richer ageing effects in glassy phenomenology. Nevertheless there are still some points that require further investigation in order to be completely understood.

First, we mention the equivalence between the asymptotic FDR $X^\infty$ defined in momentum and in real space, see equation (21). To our knowledge, the only general argument supporting this result is the one outlined in section 2.4 (originally proposed in [19]). However, a recent analysis of the dynamics of the weakly dilute Ising model [61] suggests that probably such an equivalence has a wider range of applicability. It would be desirable to have a clear proof of it and to know the conditions under which one can (or cannot) expect equation (21) to be valid, especially in connection with the possible definition of the effective temperature (see section 2.4).

Another interesting issue is the behaviour of these critical systems just at their lower critical dimensions (i.e., the dimension where the critical temperature vanishes, $d = 1$ for Ising-like systems and $d = 2$ for systems with continuous symmetries). Indeed, in passing, we mentioned [90, 94, 95] that different one-dimensional systems with Ising symmetry may have different non-equilibrium critical exponents (i.e., $\theta$). In this context it has been recently argued [53] that a quench to zero temperature at the lower critical dimension is not the dimensional continuation of a line of critical quenches in the $(T, d)$ plane (as often implicitly

assumed), but it is the continuation of a line of zero-temperature quenches, i.e., the system behaves as in the coarsening regime, although $X^{\infty} \neq 0$ (see for details [53]). Moreover the critical quench at the lower critical dimension has a very interesting aspect: being the exponent of the autocorrelation function $a - d/z + 1$ (see equation (91)) equal to zero, it is possible to write the FDR $X_{\mathbf{x}=0}$ as a function of the autocorrelation function $C_{\mathbf{x}=0}$ as it happens in more complex instances of glassy behaviour (see, e.g., [7, 8]).

Field-theoretical RG provides methods going well beyond the expansion close to the upper critical dimension that has been used so far and that we have reviewed. It will be interesting to consider different approaches like expansions close to the lower critical dimensions, perturbative expansions in fixed dimensions and non-perturbative approaches like the so-called exact RG and $1/N$-expansion. These methods might lead to more accurate theoretical estimates for the universal quantities of interest and hopefully to a more general understanding of the phenomena.

It is natural to wonder whether some of the results that have been reviewed carry over to glassy systems, which was the original motivation of the study of ageing behaviour in critical models. Recent numerical simulations of finite-dimensional ($d = 3, 4$) spin-glass models exactly at the critical point [192] show that even in this case the response and correlation functions scale according to equations (91) and (92), as in critical non-disordered system, with a non-trivial value of $X^{\infty}$. On the other hand, for a quench in the low-temperature phase a completely different behaviour is expected [8].

Last but not least, we remark that the ultimate check of all the physical theories is provided by experiments. Unfortunately, to our knowledge, only quasi-one-dimensional systems with Ising symmetry have been considered so far [193]. Although very interesting, the physics of one-dimensional systems is too peculiar ($d = 1$ is the lower critical dimension of Ising systems) to provide a test of the theoretical results currently available. Experiments in two and three dimensions are required to check the relevant aspects of ageing dynamics *at the critical point*. In principle, this could be done by considering the same experimental systems as those considered when studying phase ordering—magnetic systems, nematic-liquid crystals, binary liquids, etc [3]—exactly at their critical points.

## Acknowledgments

It is a pleasure to thank L Berthier, S Caracciolo, F Corberi, L Cugliandolo, H W Diehl, S Dietrich, M Henkel, F Krzakala, A Lefevre, E Lippiello, M Pleimling, P Sibani, A Pelissetto, F Ricci-Tersenghi, P Sollich, E Vicari and M Zannetti for stimulating discussions, correspondence and suggestions about ageing phenomena during the last two years. The authors acknowledge S Andergassen and P Sibani for a careful reading of a first version of this manuscript. PC acknowledges financial support from EPSRC grant no. GR/R83712/01.

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